Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 303 (2007) 109–134
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Moment Lyapunov exponents of a two-dimensional system under combined harmonic and real noise excitations

Wei-Chau Xie*

Department of Civil and Environmental Engineering, Faculty of Engineering, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 8 July 2005; received in revised form 25 October 2006; accepted 21 December 2006
Available online 19 March 2007

Abstract

The moment Lyapunov exponents and the Lyapunov exponents of a two-dimensional system under combined excitations of harmonic and real noise, which is modelled as an Ornstein–Uhlenbeck process, are studied. The moment Lyapunov exponents and the Lyapunov exponents are important characteristics determining the moment and almost-sure stability of a stochastic dynamical system. The eigenvalue problem governing the moment Lyapunov exponent is established. A regular perturbation method is applied to solve the eigenvalue problem to obtain second-order, weak noise expansions of the moment Lyapunov exponents. The influence of the real noise excitation on the parametric resonance due to the harmonic excitation is investigated.

1. Introduction

The investigation of the dynamic stability of elastic systems, such as slender columns and thin plates under axial loading, or buildings, bridges, and aircraft structures under wind loading, frequently leads to the study of the dynamical behaviour of the solutions of a parametrized family of differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \gamma), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n, \quad \mathbf{f}(\mathbf{0}, \gamma) = \mathbf{0}, \tag{1}$$

where γ is a scalar parameter characterizing the load condition.

In many practical situations, the loading may be subjected to fluctuations of a stochastic nature, such as those arising from earthquakes, wind, and ocean waves. White noise has been widely used in engineering applications for the modelling of noise processes because of its simplicity and the availability of rigorous mathematical theory. However, white-noise does not exist as a physically realizable process and the singular behaviour it exhibits does not arise in any realizable context. On the other hand, the Ornstein–Uhlenbeck process is a simple, Gaussian, explicitly representable stationary process, and is often used to model a realizable noise process. The Ornstein–Uhlenbeck process is given by $\mathrm{d}\xi(t) = -\alpha\xi(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$, where $W(t)$ is a standard Wiener process. Letting $\sigma = \sqrt{2D}\alpha$, the correlation function and spectral density of the Ornstein–Uhlenbeck process $\xi(t)$ are $R(\tau) = D\alpha \mathrm{e}^{-\alpha|\tau|}$ and

*Tel.: +1 519 888 4567x33988; fax: +1 519 888 6197.
E-mail address: xie@uwaterloo.ca.

0022-460X/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2006.12.030

$S(\omega) = D[1+(\omega/\alpha)^2]^{-1}$, in which the parameter α characterizes the bandwidth of the noise and D is related to the spectral density of the noise. When $\alpha = \sigma \to \infty$, the Ornstein–Uhlenbeck process $\xi(t)$ approaches $\sqrt{2D}\dot{W}(t)$, where $\dot{W}(t)$ denotes formally a unit Gaussian white-noise process. Hence, if one sets $D = \frac{1}{2}$, $\alpha = \sigma \to \infty$, the Ornstein–Uhlenbeck process $\xi(t)$ approaches the unit Gaussian white-noise process $\dot{W}(t)$.

There are engineering situations where the loading on the system contains both periodic components and stochastic fluctuations. An example of such a system is the uncoupled flapping motion of rotor blades in forward flight under the effect of atmospheric turbulence.

The sample or almost-sure stability of the trivial solution of system, described by Eq. (1) is determined by the Lyapunov exponent of its linearized system

$$\dot{\mathbf{x}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\gamma)|_{\mathbf{x}=\mathbf{0}}\mathbf{x}. \tag{2}$$

The Lyapunov exponents, which are deterministic numbers, characterize the average exponential rates of growth of the solutions of the system when the time parameter t is large, and is defined by

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\log\|\mathbf{x}(t)\|, \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. In Eq. (3), $\|\mathbf{x}(t)\|$ is a stochastic process and the limit exists almost-surely or with probability one (w.p.1). In other words, the almost-sure convergence guarantees that the limit given by Eq. (3) exists except for a set of solutions $\mathbf{x}(t)$ with probability zero. Depending on the initial conditions $\mathbf{x}(0)$, there are n Lyapunov exponents for the system described by Eq. (2). The trivial solution of the dynamical system is stable with probability one if the top Lyapunov exponent is negative, whereas it is unstable with probability one if the top Lyapunov exponent is positive.

On the other hand, the stability of the pth moment, $\mathsf{E}[\|\mathbf{x}(t)\|^p]$, of the trivial solution of the dynamical system is determined by the pth moment Lyapunov exponent, a deterministic number, defined by

$$\Lambda(p) = \lim_{t\to\infty}\frac{1}{t}\log\,\mathsf{E}[\|\mathbf{x}(t)\|^p], \tag{4}$$

where $\mathsf{E}[\cdot]$ denotes expected value. Eq. (4) defines the Lyapunov exponent of the pth moment $\mathsf{E}[\|\mathbf{x}(t)\|^p]$, which is a deterministic function of t, rather than a stochastic process; the convergence of the limit is thus understood in the ordinary sense for deterministic functions. If $\Lambda(p)<0$, then $\mathsf{E}[\|\mathbf{x}(t)\|^p] \to 0$ as $t\to\infty$. The pth moment Lyapunov exponent $\Lambda(p)$ is a convex analytic function in p with $\Lambda(0)=0$ and $\Lambda'(0)$ is equal to the top Lyapunov exponent λ. The non-trivial zero δ of $\Lambda(p)$, i.e. $\Lambda(\delta)=0$, is called the stability index.

To have a complete picture of the dynamic stability of a dynamical system, it is important to study both the sample and moment stability and to determine both the top Lyapunov exponent and the pth moment Lyapunov exponent. A systematic presentation of the theory of random dynamical systems and a comprehensive list of references can be found in Arnold [1].

Although the moment Lyapunov exponents are important in the study of dynamic stability of stochastic systems and it is relatively straightforward to set up the partial differential eigenvalue problems governing the moment Lyapunov exponents (see, e.g. Refs. [2,3]), the actual evaluations of the moment Lyapunov exponents are very difficult. In the last decade, researchers have attempted to devise various approaches to obtain approximate results of the moment Lyapunov exponents. However, only a few results have been published so far. Using the analytic property of the moment Lyapunov exponents, Arnold et al. [4] obtained weak noise expansions of the moment Lyapunov exponents of a two-dimensional system in terms of εp, where ε is a small parameter, under both white-noise and real noise excitations. Khasminskii and Moshchuk [5] obtained an asymptotic expansion of the moment Lyapunov exponent of a two-dimensional system under white-noise parametric excitation in terms of the small fluctuation parameter ε, from which the stability index was obtained. Sri Namachchivaya and Vedula [6] obtained a general asymptotic approximation for the moment Lyapunov exponent and the Lyapunov exponent for a four-dimensional system with one critical mode and another asymptotically stable mode driven by a small intensity stochastic process. Sri Namachchivaya and Van Roessel [7] studied the moment Lyapunov exponents of two coupled oscillators driven by real noise. Xie obtained weak noise expansions of the moment Lyapunov exponent, the Lyapunov exponent, and the stability index, in terms of the small fluctuation parameter, of a two-dimensional system exhibiting pitch-fork bifurcation under real noise excitation [8] and under bounded noise excitation [9].

The almost-sure stability of dynamical systems under combined harmonic and stochastic excitations was first studied by Sri Namachchivaya [10] by evaluating the largest Lyapunov exponents. Weak noise expansions of the moment Lyapunov exponents and Lyapunov exponents of two-dimensional systems under both harmonic and white-noise excitations were studied by Baxendale [11] and Xie [12].

In this paper, a two-dimensional system under combined parametric excitations of harmonic and real noise, which is a more realistic model of noises in engineering applications, is studied. Weak noise expansions of the moment Lyapunov exponents and Lyapunov exponents are obtained. The effect of the real noise excitation on the parametric resonance due to the harmonic excitation is investigated. Because of the extra harmonic excitation in the stochastic equation of motion, the eigenvalue problem governing the moment Lyapunov exponent is a second-order partial differential equation with *three* independent variables. The perturbation approach used in solving the eigenvalue problem is more complex and significantly different from that applied in Ref. [8], in which the moment Lyapunov exponent is governed by a second-order partial differential eigenvalue problem with only *two* independent variables.

2. Formulation

2.1. Equations of motion

Consider a two-dimensional system under combined harmonic and real noise excitations as follows:

$$q''(\tau) + 2\varepsilon\beta q'(\tau) + [\Omega_0^2 + \varepsilon\mu_0 \sin \hat{v}_0\tau + \varepsilon^\rho \gamma_0 \xi(\tau)]q(\tau) = 0, \tag{5}$$

where $\varepsilon > 0$ is a small parameter, ρ is a proper scaling parameter to be determined so that the contributions of the harmonic and real noise excitations are comparable, and $\xi(\tau)$ is a real noise process in time τ, modelled as an Ornstein–Uhlenbeck process, given by

$$\mathrm{d}\xi(\tau) = -\alpha_0 \xi(\tau)\,\mathrm{d}\tau + \sigma_0\,\mathrm{d}W(\tau), \tag{6}$$

in which $W(\tau)$ is a standard Wiener process in time τ.

The damping term in Eq. (5) can be removed by applying the transformation $q(\tau) = x(\tau)\mathrm{e}^{-\varepsilon\beta\tau}$ to yield

$$x''(\tau) + [\Omega^2 + \varepsilon\mu_0 \sin \hat{v}_0\tau + \varepsilon^\rho \gamma_0 \xi(\tau)]x(\tau) = 0, \tag{7}$$

where $\Omega^2 = \Omega_0^2 - \varepsilon^2\beta^2$. Eq. (6) can be further simplified by applying the time scaling $t = \Omega\tau$. The time scaling reduces Eqs. (7) and (6) to

$$\ddot{x}(t) + [1 + \varepsilon\mu \sin vt + \varepsilon^\rho \gamma \zeta(t)]x(t) = 0, \tag{8}$$

$$\mathrm{d}\zeta(t) = -\alpha\zeta(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \tag{9}$$

where

$$\mu = \frac{\mu_0}{\Omega}, \quad v = \frac{\hat{v}_0}{\Omega}, \quad \gamma = \frac{\gamma_0}{\Omega^2}, \quad \alpha = \frac{\alpha_0}{\Omega}, \quad \sigma = \frac{\sigma_0}{\sqrt{\Omega}},$$

and $W(t)$ is the standard Wiener process in time t.

In the absence of the real noise excitation, i.e. when $\gamma = 0$, Eq. (8) reduces to the Mathieu's equation

$$\ddot{x}(t) + (1 + \varepsilon\mu \sin vt)x(t) = 0.$$

The nth-order parametric resonance occurs when the frequency of the sinusoidal excitation v is in the vicinities of $2/n$, $n = 1, 2, \ldots$. When $v \approx 2$, the primary resonance occurs. The first-order approximation of the instability region can be obtained using the method of averaging or the method of multiple scales and is given by (see, e.g. Ref. [13])

$$2 - \tfrac{1}{2}(\varepsilon\mu) \leqslant v \leqslant 2 + \tfrac{1}{2}(\varepsilon\mu).$$

For a given value of μ, the width of the primary instability region is $\varepsilon\mu$, i.e. of the order ε. In the instability region, the trivial solution $(x, \dot{x}) = (0, 0)$ becomes unstable and grows exponentially. When $v \approx 1$, the secondary resonance occurs and the first-order approximation of the instability region is given by

$$1 - \tfrac{5}{24}(\varepsilon\mu)^2 \leqslant v \leqslant 1 + \tfrac{1}{24}(\varepsilon\mu)^2.$$

For a given value of μ, the width of the secondary instability region is $\frac{1}{4}(\varepsilon\mu)^2$, which is of the order ε^2.

On the other hand, when the amplitude of the sinusoidal excitation μ is zero, Eq. (8) is a single degree-of-freedom system under real noise parametric excitation

$$\ddot{x}(t) + [1 + \varepsilon^{\rho}\gamma\zeta(t)]x(t) = 0.$$

From the method of stochastic averaging [13–15], it is well known that for the noise to have a contribution of the order of ε, the noise should be of the order $\varepsilon^{1/2}$, i.e. $\rho = 1/2$.

When studying the effect of noise on the parametric resonance of Mathieu's equation, it is important to properly scale the noise so that the effects of the noise and the sinusoidal excitations are comparable. If the noise is too weak, it has no effect on the parametric resonance in the first-order approximation. On the other hand, if the noise is too strong, it overpowers the sinusoidal excitation and the effect of parametric resonance is of a higher order.

For the primary resonance, the effect of the sinusoidal excitation is of the order ε. Hence, the order of the real noise is taken as $\varepsilon^{1/2}$, i.e. $\rho = 1/2$. For the secondary resonance, the effect of the sinusoidal excitation is of the order ε^2, which leads to that the order of the real noise must be ε, i.e. $\rho = 1$, to have a comparable contribution.

2.2. Establishing the eigenvalue problem governing the moment Lyapunov exponents

The moment and almost-sure stability of the system described by Eq. (8) are determined by the moment Lyapunov exponent and the top Lyapunov exponent, respectively. The Lyapunov exponents and the moment Lyapunov exponents of the systems described in Eqs. (5), (7), and (8) are related by the following relationships:

$$\lambda_{q(\tau)} = -\varepsilon\beta + \lambda_{x(\tau)} = -\varepsilon\beta + \Omega\lambda_{x(t)},$$

$$\Lambda_{q(\tau)}(p) = -\varepsilon p\beta + \Lambda_{x(\tau)}(p) = -\varepsilon p\beta + \Omega\Lambda_{x(t)}(p).$$

The moment Lyapunov exponent of system of Eq. (8) is the eigenvalue of a eigenvalue problem. In this section, two approaches are presented to establish the partial differential eigenvalue problem governing the pth moment Lyapunov exponent.

The first approach uses the theory of stochastic dynamical system [2]. Denoting $\theta = vt$, θ may be considered as a random process with generator $G_\theta = v\partial/\partial\theta$. The real noise $\zeta(t)$ defined by Eq. (9) has the generator

$$G_\zeta = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} - \alpha\zeta\frac{\partial}{\partial\zeta}.$$

Letting $x_1 = x$, $x_2 = \dot{x}$, Eq. (8) may be written in the form of state equations

$$\mathrm{d}\mathbf{x} = \mathbf{B}_0\mathbf{x}\,\mathrm{d}t + \mathbf{B}_1\mathbf{x}\,\mathrm{d}W, \tag{10}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} 0 & 1 \\ -(1 + \varepsilon\mu\sin\theta + \varepsilon^{\rho}\gamma\zeta) & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Applying the Khasminskii transformation [16]:

$$s_1 = \frac{x_1}{\|\mathbf{x}\|} = \cos\varphi, \quad s_2 = \frac{x_2}{\|\mathbf{x}\|} = \sin\varphi, \quad \mathbf{s} = \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}, \quad \hat{\mathbf{s}} = \begin{Bmatrix} \sin\varphi \\ -\cos\varphi \end{Bmatrix}, \tag{11}$$

where $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ is the Euclidean norm of vector $\mathbf{x}$, the random process $\mathbf{x}$ is projected onto the unit sphere $\mathbf{s}$. The moment Lyapunov exponent $\Lambda_{x(t)}(p)$ of the system described by Eqs. (8), or Eq. (10) in the form of state equations, is the principal eigenvalue of the eigenvalue problem [2]

$$\mathscr{L}(p)T(\zeta,\varphi,\theta) = \Lambda_{x(t)}(p)T(\zeta,\varphi,\theta), \tag{12}$$

where $\mathscr{L}(p) = G_\zeta + G_\theta + L + pX + pQ + \frac{1}{2}p^2 R$.

To evaluate L, X, Q, and R, it is necessary to determine β_i, h_i, and q_i, for $i = 0$ and 1, which are given by

$$\beta_i = \hat{\mathbf{s}}^{\mathrm{T}}\mathbf{B}_i\mathbf{s}, \quad \beta_0 = 1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi, \quad \beta_1 = 0,$$

$$h_i = -\beta_i\frac{\partial}{\partial\varphi}, \quad h_0 = -[1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi]\frac{\partial}{\partial\varphi}, \quad h_1 = 0,$$

$$q_i = \mathbf{s}^{\mathrm{T}}\mathbf{B}_i\mathbf{s}, \quad q_0 = -(\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos\varphi\sin\varphi, \quad q_1 = 0.$$

Hence,

$$L = h_0 + \tfrac{1}{2}h_1^2 = -[1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi]\frac{\partial}{\partial\varphi},$$

$$X = q_1 h_1 = 0,$$

$$Q = q_0 - q_1^2 + \tfrac{1}{2}\mathbf{s}^{\mathrm{T}}(\mathbf{B}_1 + \mathbf{B}_1^{\mathrm{T}})\mathbf{B}_1\mathbf{s} = -(\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos\varphi\sin\varphi,$$

$$R = q_1^2 = 0.$$

Therefore, the infinitesimal differential operator $\mathscr{L}(p)$ is given by

$$\mathscr{L}(p) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\zeta^2} - \alpha\zeta\frac{\partial}{\partial\zeta} + v\frac{\partial}{\partial\theta} - [1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi]\frac{\partial}{\partial\varphi} - p(\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos\varphi\sin\varphi. \tag{13}$$

The eigenvalue problem (Eq. (12)) with the infinitesimal differential operator given by Eq. (13) can also be derived by using a more intuitive approach. This second approach was originally employed by Wedig [3] to derive the eigenvalue problem for the moment Lyapunov exponent of a two-dimensional linear Itô stochastic system.

Rewrite Eqs. (8) and (9) as a four-dimensional system as

$$\mathrm{d}\begin{Bmatrix} x_1 \\ x_2 \\ \zeta \\ \theta \end{Bmatrix} = \begin{Bmatrix} x_2 \\ -(1 + \varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)x_1 \\ -\alpha\zeta \\ v \end{Bmatrix}\mathrm{d}t + \begin{Bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{Bmatrix}\mathrm{d}W. \tag{14}$$

Employing the Khasminskii transformation shown in Eq. (11) and defining the pth norm $P = \|\mathbf{x}\|^p$, the Itô differential equations for the pth norm P and the angle φ can be obtained by using Itô's Lemma:

$$\mathrm{d}\begin{Bmatrix} P \\ \varphi \\ \zeta \\ \theta \end{Bmatrix} = \begin{Bmatrix} -pP(\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos\varphi\sin\varphi \\ -[1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi] \\ -\alpha\zeta \\ v \end{Bmatrix}\mathrm{d}t + \begin{Bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{Bmatrix}\mathrm{d}W. \tag{15}$$

Applying a linear transformation

$$S = T(\zeta,\varphi,\theta)P, \quad P = T^{-1}(\zeta,\varphi,\theta)S, \quad 0 \leqslant \varphi < \pi, \ 0 \leqslant \theta < 2\pi,$$

the Itô differential equation for the transformed pth norm process S can also be obtained by using Itô's Lemma:

$$\mathrm{d}S = \{\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} - \alpha\zeta T_\zeta + \nu T_\theta - [1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi]T_\varphi \\ - p(\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos\varphi\sin\varphi T\}P\,\mathrm{d}t + \sigma T_\zeta P\,\mathrm{d}W. \tag{16}$$

For a bounded and non-singular transformation $T(\zeta,\varphi,\theta)$, both processes P and S are expected to have the same stability behaviour. Therefore, $T(\zeta,\varphi,\theta)$ is chosen so that the drift term of the Itô differential given in Eq. (16) is independent of the processes ζ, φ, and θ, i.e.,

$$\mathrm{d}S = \Lambda S\,\mathrm{d}t + \sigma T_\zeta T^{-1} S\,\mathrm{d}W. \tag{17}$$

Comparing the drift terms in Eqs. (16) and (17), it is seen that such a transformation $T(\zeta,\varphi,\theta)$ is the eigenfunction of the eigenvalue problem

$$\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} - \alpha\zeta T_\zeta + \nu T_\theta - [1 + (\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos^2\varphi]T_\varphi \\ - p(\varepsilon\mu\sin\theta + \varepsilon^\rho\gamma\zeta)\cos\varphi\sin\varphi T = \Lambda T \tag{18}$$

with Λ being the eigenvalue.

From Eq. (17), it is clear that Λ is the Lyapunov exponent of the transformed pth norm process S; hence Λ is the moment Lyapunov exponent of system (14) or (8), i.e. $\Lambda = \Lambda_{x(t)}(p)$. By comparing Eqs. (12) and (13) with Eq. (18), it is seen that the eigenvalue problem governing the moment Lyapunov exponent $\Lambda = \Lambda_{x(t)}(p)$, derived by using the general theory of moment Lyapunov exponent [2], is the same as that derived by using a more intuitive approach originally employed by Wedig [3].

3. Moment Lyapunov exponents

As mentioned earlier, in the absence of the real noise excitation, the system described by Eq. (8) is in parametric resonance when ν is in the vicinity of $2, 1, \frac{2}{3}, \ldots$. In this paper, the effect of real noise on the primary and the secondary parametric resonance is studied through the determination of the moment Lyapunov exponent. Let the harmonic excitation frequency $\nu = \nu_0 + \varepsilon^n\Delta, n = 1, 2,$ where $\nu_0 = 2/n$ is the harmonic excitation frequency corresponding to a parametric resonance of order n in the absence of the real noise excitation, and Δ is the mistune parameter.

In this section, the stochastic stability of the system described by Eq. (8) is studied through the determination of the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ by solving the eigenvalue problem (Eqs. (12) or (18)). A method of regular perturbation (see, e.g. Ref. [17]) is applied to obtain weak noise expansions of the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ and the Lyapunov exponent $\lambda_{x(t)}$.

3.1. Primary parametric resonance, $n = 1$, $\rho = \frac{1}{2}$, and $\nu_{0=2}$

3.1.1. Method of regular perturbation

In the absence of the real noise excitation, the system described in Eq. (8) is in primary parametric resonance when $\nu = \nu_0 + \varepsilon\Delta$, $\nu_0 = 2$. Applying the transformation $\theta = \psi - 2\varphi$, the infinitesimal differential operator in Eq. (12) becomes

$$\mathscr{L}(p) = L_0 + \varepsilon^{1/2}L_1 + \varepsilon L_2, \tag{19}$$

where

$$L_0 = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} - \alpha\zeta\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\varphi},$$

$$L_1 = -\gamma\zeta\left[\cos^2\varphi\left(2\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\varphi}\right) + p\cos\varphi\sin\varphi\right],$$

$$L_2 = \Delta\frac{\partial}{\partial\psi} - \mu\sin(\psi - 2\varphi)\left[\cos^2\varphi\left(2\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\varphi}\right) + p\cos\varphi\sin\varphi\right].$$

Expand the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ and the eigenfunction $T(\zeta,\varphi,\psi)$:

$$\Lambda_{x(t)}(p) = \sum_{i=0}^{\infty}\varepsilon^{i/2}\Lambda_i(p), \quad T(\zeta,\varphi,\psi) = \sum_{i=0}^{\infty}\varepsilon^{i/2}T_i(\zeta,\varphi,\psi), \tag{20}$$

where $T_i(\zeta,\varphi,\psi)$ are periodic functions in φ of period π and in ψ of period 2π. Substituting Eq. (20) into Eq. (19), expanding, and equating terms of equal power of ε, results in the perturbation equations,

$$\begin{aligned}
&\text{Zeroth-order:} && L_0T_0 = \Lambda_0T_0,\\
&\text{First-order:} && L_0T_1 + L_1T_0 = \Lambda_1T_0 + \Lambda_0T_1,\\
&\text{Second-order:} && L_0T_2 + L_1T_1 + L_2T_0 = \Lambda_2T_0 + \Lambda_1T_1 + \Lambda_0T_2,\\
&\text{Third-order:} && L_0T_3 + L_1T_2 + L_2T_1 = \Lambda_3T_0 + \Lambda_2T_1 + \Lambda_1T_2 + \Lambda_0T_3,\\
&\ \vdots && \ \vdots
\end{aligned} \tag{21}$$

By solving the zeroth-order, first-order, and second-order perturbation equations in Eq. (21), it is shown in Appendix A that $\Lambda_0(p) = \Lambda_1(p) = 0$ and $\Lambda_2(p)$ is given by the eigenvalue of the second-order ordinary differential eigenvalue problem (Eq. (A.11)), i.e.

$$a\ddot{\Psi}_0(\psi) + (b + 2q\sin\psi)\dot{\Psi}_0(\psi) + [\Lambda_2 + cp(p+2) - pq\cos\psi]\Psi_0(\psi) = 0, \tag{22}$$

where the coefficients a, b, c, and q are given in Appendix A.

3.1.2. Determination of Λ_2

The second-order perturbation of the moment Lyapunov exponent Λ_2 can be obtained by solving the eigenvalue problem given in Eq. (22). Since the coefficients of Eq. (22) are periodic functions of period 2π, a series expansion of the eigenfunction $\Psi_0(\psi)$ may be taken:

$$\Psi_0(\psi) = C_0 + \sum_{k=1}^{N}(C_k\cos k\psi + S_k\sin k\psi), \tag{23}$$

where C_0, C_k, S_k, $k = 1,2,\ldots,N$, are constants to be determined. Substituting Eq. (23) into Eq. (22), multiplying the resulting equation by $\cos m\psi$, $\sin m\psi$, $m = 0,1,\ldots,N$, respectively, and integrating with respect to ψ from 0 to 2π results in a set of $2N+1$ homogeneous linear equations for the unknown coefficients C_0, C_k, S_k, $k = 1,2,\ldots,N$. In order to have a non-trivial solution, the determinant of the coefficient matrix of these $2N+1$ linear algebraic equations $\Delta^{(N)}$ must be zero. The determinantal equation $\Delta^{(N)}$ leads to a polynomial equation of degree $2N+1$ in Λ_2 of the form

$$\Lambda_2^{2N+1} + d_{2N}\Lambda_2^{2N} + \cdots + d_1\Lambda_2 + d_0 = 0. \tag{24}$$

An approximate result of Λ_2 can be obtained by solving Eq. (24). Note that the exact result of Λ_2 is obtained when the number of terms $N \to \infty$.

When $N = 1$, Eq. (24) is a cubic equation, the coefficients of which are

$$\begin{aligned}
d_0 = {} & c^3p^6 + 6c^3p^5 - (2ac^2 - 12c^3 + \tfrac{1}{2}cq^2)p^4 - (8ac^2 - 8c^3 + 2cq^2)p^3\\
& + (\tfrac{1}{2}aq^2 - 8ac^2 - 2cq^2 + b^2c + a^2c)p^2 + (2b^2c + aq^2 + 2a^2c)p,
\end{aligned}$$

$$d_1 = 3c^2p^4 + 12c^2p^3 - (4ac - 12c^2 + \tfrac{1}{2}q^2)p^2 - (8ac + q^2)p + a^2 + b^2,$$

$$d_2 = 3cp^2 + 6cp - 2a.$$

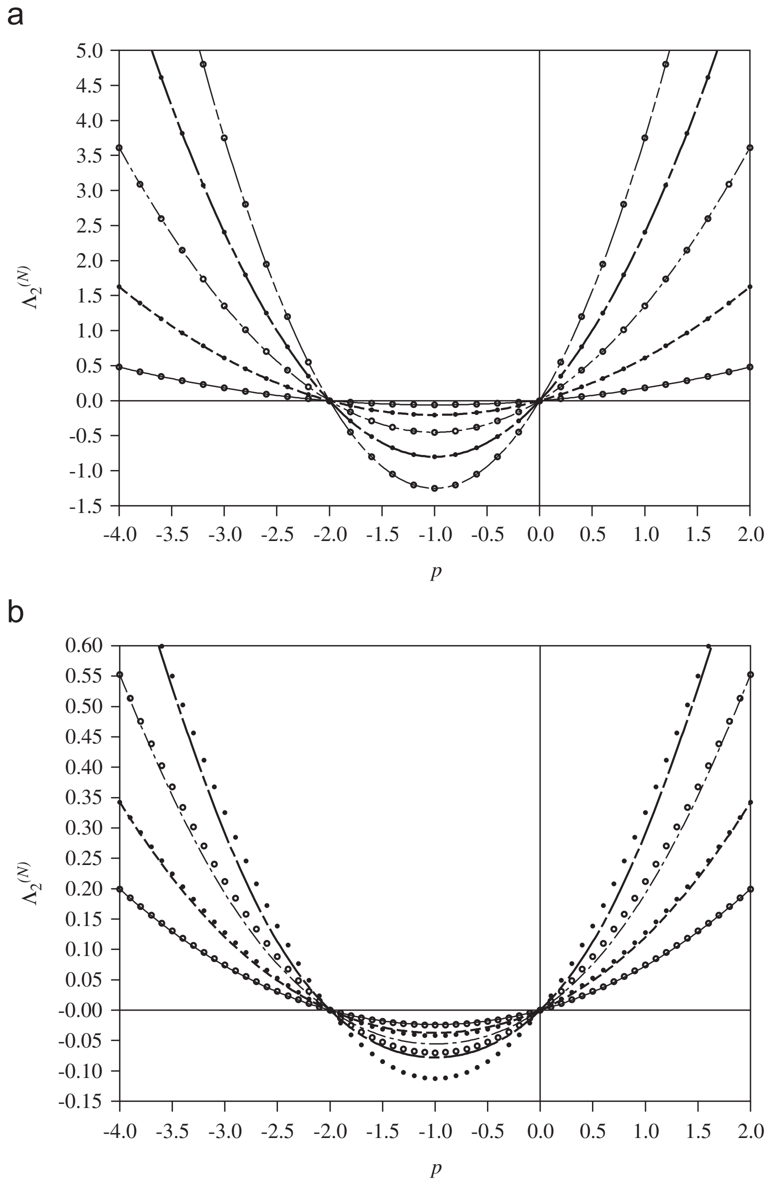

Fig. 1. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(N)}$, primary resonance, $\Delta = 1.0$, $\gamma = 1.0$, $\alpha = 1.0$. Lines, $N = 1$, analytical results; dots (○ or ●), $N = 8$, numerical results: (a) $\mu = 1.0$; —, $\sigma = 2.0$; - - - , $\sigma = 4.0$; ·— · —·, $\sigma = 6.0$; -——-, $\sigma = 8.0$; and - -——- -, $\sigma = 10.0$; (b) $\sigma = 1.0$; —, $\mu = 1.0$; - - -, $\mu = 1.5$; ·— · —·, $\mu = 2.0$; and -——-, $\mu = 2.5$.

The solution of Eq. (24) with $N = 1$ is given by

$$\Lambda_2^{(1)} = \frac{1}{6}(A_2 - 2d_2) - \frac{2}{3}\frac{3d_1 - d_2^2}{A_2}, \tag{25}$$

where

$$A_2 = (-108d_0 + 36d_1d_2 - 8d_2^3 + 12A_1)^{1/3},$$
$$A_1 = (81d_0^2 - 54d_0d_1d_2 + 12d_1^3 + 12d_0d_2^3 - 3d_1^2d_2^2)^{1/2}.$$

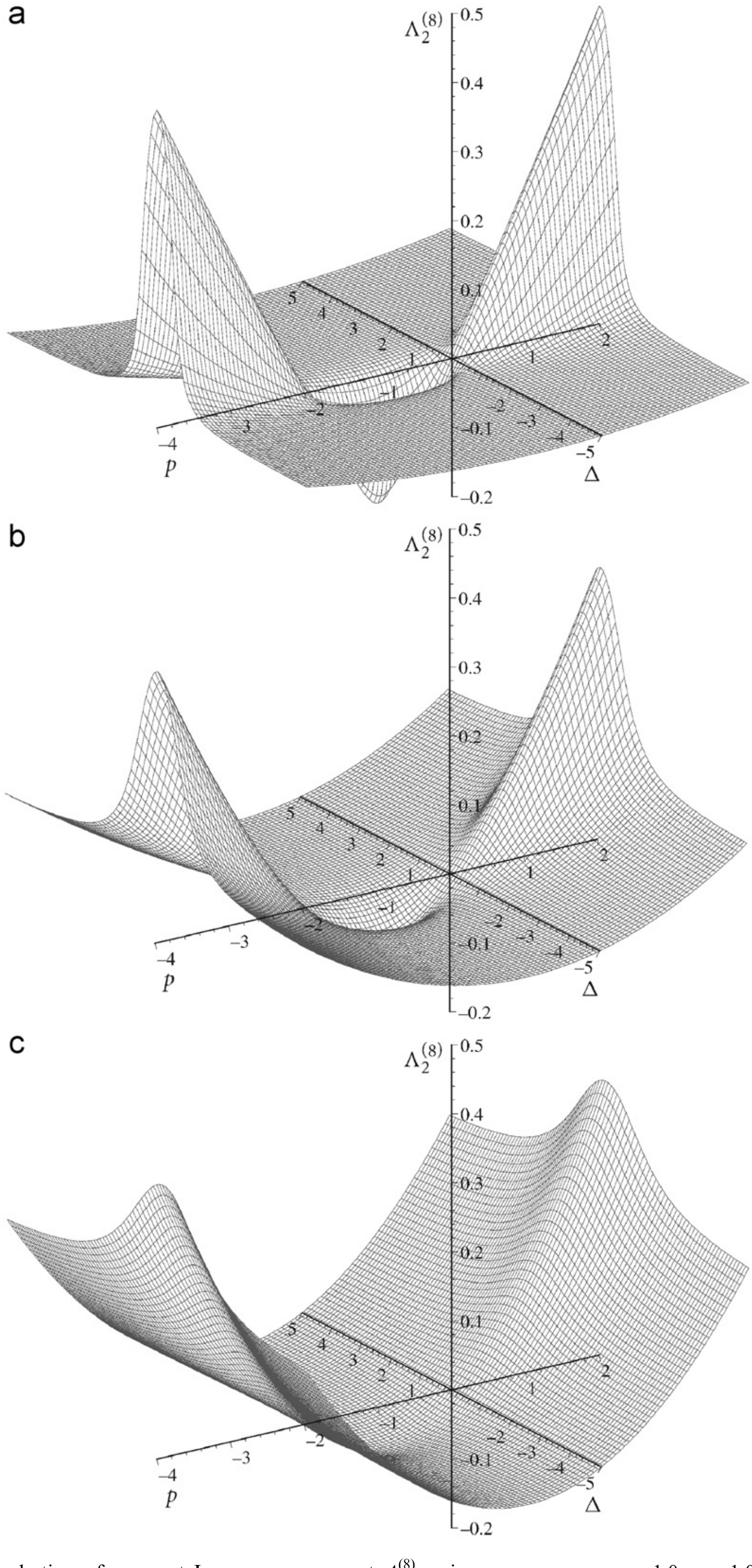

Fig. 2. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(8)}$, primary resonance, $\mu = 1.0$, $\gamma = 1.0$ and $\alpha = 1.0$. (a) $\sigma = 0.5$; (b) $\sigma = 1.0$; and (c) $\sigma = 1.5$.

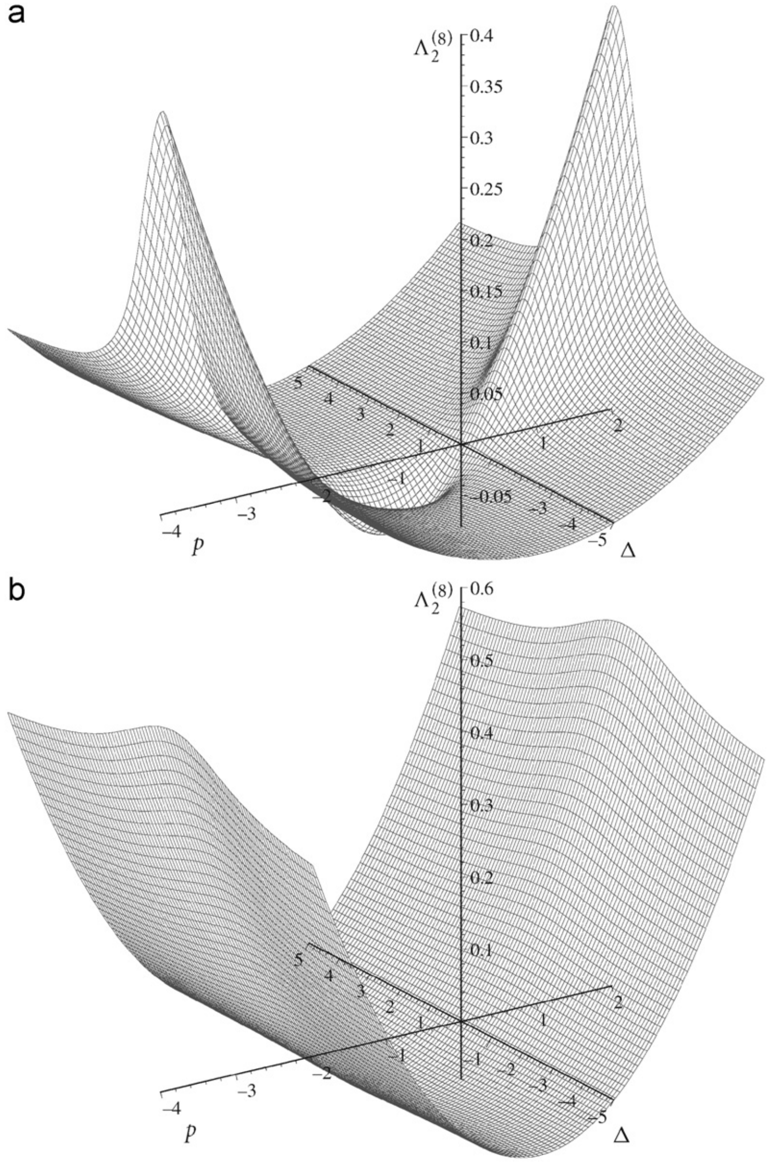

Fig. 3. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(8)}$, primary resonance, $\mu = 1.0$, $\gamma = 2.0$ and $\alpha = 1.0$. (a) $\sigma = 0.5$ and (b) $\sigma = 1.0$.

When $N > 1$, there is no analytical solution available for Eq. (24) and numerical approach has to be employed to find an approximate root $\Lambda_2^{(N)}$. In Fig. 1, the analytical result $\Lambda_2^{(1)}$ is plotted along with the numerical result $\Lambda_2^{(8)}$. It is seen that, for a given value of $\mu = 1$, the two results agree extremely well for all values of σ. On the other hand, for a given value of $\sigma = 1$, $\Lambda_2^{(1)}$ and $\Lambda_2^{(8)}$ agree very well for smaller values of μ up to 2. When the value of μ increases, discrepancies exist between $\Lambda_2^{(1)}$ and $\Lambda_2^{(8)}$. Three-dimensional surface plots of $\Lambda_2^{(8)}$ are shown in Fig. 2 for $\mu = 1.0$, $\gamma = 1.0$, $\alpha = 1.0$, and $\sigma = 0.5$, 1.0, and 1.5. As mentioned earlier, the system described by Eq. (8) is in primary parametric resonance in the absence of the real noise excitation. To study the influence of the real noise excitation and the harmonic excitation on the parametric resonance and the moment Lyapunov exponent, $\Lambda_2^{(8)}$ are shown in Fig. 3 for $\mu = 1.0$, $\gamma = 2.0$, $\alpha = 1.0$, and $\sigma = 0.5$ and 1.0, and in Fig. 4 for $\mu = 2.0$,$\gamma = 1.0$, $\alpha = 1.0$, and $\sigma = 0.5$ and 1.0.

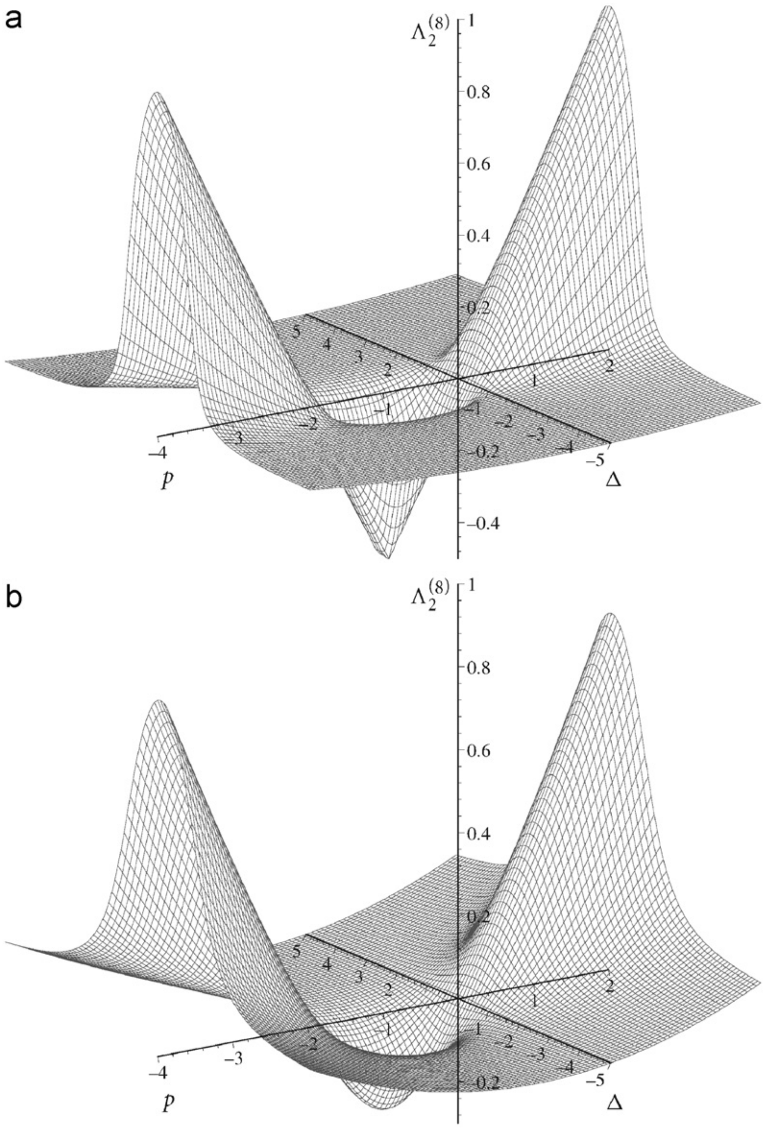

Fig. 4. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(8)}$, primary resonance, $\mu = 2.0$, $\gamma = 1.0$ and $\alpha = 1.0$. (a) $\sigma = 0.5$ and (b) $\sigma = 1.0$.

It is noted that the use of truncated Fourier series given in Eq. (23) with relatively small values of N results in accurate approximations of the moment Lyapunov exponents for the system of Eq. (8). However, no general statement can be made beyond the system considered in this paper. When an approximate value $\Lambda_2^{(N)}$ is obtained from Eq. (24), an approximation of the moment Lyapunov exponent is given by $\Lambda_{x(t)}(p) \approx \varepsilon\Lambda_2^{(N)} + o(\varepsilon)$.

It is seen that for small values of the noise intensity σ, small values of the amplitude of the real noise excitation γ, or large values of the amplitude of the harmonic excitation μ, the influence of the real noise excitation is small and the harmonic excitation is dominant; hence the effect of the primary parametric resonance is significant. On the other hand, when the noise intensity σ is increased, the amplitude of the real noise excitation γ is large, or the amplitude of the harmonic excitation μ is small, the impact of the real noise excitation intensifies and the prominence of the primary parametric resonance is decreased.

Furthermore, note that when $\Delta \to \pm\infty$, the effect of parametric resonance diminishes and the influence of the harmonic excitation vanishes. Obviously, when $\mu \to 0$, the influence of the harmonic excitation diminishes.

By taking limits of Eq. (25) as $\Delta \to \pm\infty$ or $\mu \to 0$, it can be easily shown that

$$\lim_{\Delta\to\pm\infty} \Lambda_2^{(1)} = \lim_{\mu\to 0} \Lambda_2^{(1)} = \frac{p(p+2)\gamma^2\sigma^2}{16(\alpha^2+4)}, \tag{26}$$

which is the same as the second-order perturbation of the moment Lyapunov exponent of a two-dimensional system under real noise excitation (Eq. (39) in Ref. [8]).

As mentioned earlier, taking $\sigma = \alpha \to \infty$, the real noise, modelled by an Ornstein–Uhlenbeck process, approaches a unit Gaussian white-noise process. By setting $\mu = 0$, the system of Eq. (8) becomes a two-dimensional system under white-noise excitation

$$\begin{aligned} \mathrm{d}x_1 &= x_2\,\mathrm{d}t, \\ \mathrm{d}x_2 &= -x_1\,\mathrm{d}t - \varepsilon^{1/2}\gamma x_1\,\mathrm{d}W(t). \end{aligned} \tag{27}$$

From Eq. (26), one obtains the moment Lyapunov exponent of the system described by Eq. (27)

$$\lim_{\substack{\mu\to 0\\ \sigma=\alpha\to\infty}} \Lambda_{x(t)}(p) \approx \varepsilon \lim_{\substack{\mu\to 0\\ \sigma=\alpha\to\infty}} \Lambda_2^{(N)} + o(\varepsilon) = \varepsilon\frac{p(p+2)\gamma^2}{16} + o(\varepsilon),$$

which can be easily shown to be the same as the second-order approximation of the moment Lyapunov exponent of system, described by Eq. (27), obtained by Khasminskii and Moshchuk (Eq. (48) in Ref. [5]).

3.1.3. Determination of λ_2

It is well known that the Lyapunov exponent is related to the moment Lyapunov exponent by

$$\lambda_{x(t)} = \lim_{p\to 0}\frac{\Lambda_{x(t)}(p)}{p} \approx \varepsilon\lambda_2^{(N)} + o(\varepsilon), \quad \lambda_2^{(N)} = \lim_{p\to 0}\frac{\Lambda_2^{(N)}(p)}{p}. \tag{28}$$

From Eq. (28) it can be inferred that $\Lambda_2^{(N)} = O(p)$ when $p \to 0$, and hence $[\Lambda_2^{(N)}]^k = o(p)$ for $k > 1$. From Eq. (24), the second-order perturbation of the Lyapunov exponent is given by

$$\lambda_2^{(N)} = \lim_{p\to 0}\frac{d_0}{d_1}. \tag{29}$$

Since the determination of $\lambda_2^{(N)}$ does not require the solution of the polynomial in Eq. (24), a larger value of N can be taken. When $N = 8$, $\lambda_2^{(8)}$ is given by $\lambda_2^{(8)} = -N^{(8)}/D^{(8)}$, where $N^{(8)}$ and $D^{(8)}$ are given in Appendix B. The expressions of $\lambda_2^{(N)}$ for larger values of N are also obtained but are not presented here due to the limitation in the length of the paper.

When $\Delta \to \pm\infty$, the effect of the harmonic excitation diminishes and the influence of the real noise excitation dominates. By taking the indicated limits, it can be shown that, for all values of N,

$$\lim_{\Delta\to\pm\infty} \lambda_2^{(N)} = \lim_{\mu\to 0} \lambda_2^{(N)} = \frac{\gamma^2\sigma^2}{8(\alpha^2+4)}, \tag{30}$$

which is the same as the second-order perturbation of the Lyapunov exponent of a two-dimensional system under real noise excitation (Eq. (52) in Xie [8]).

From Eq. (30), the Lyapunov exponent for the system described by (27) is

$$\lim_{\substack{\mu\to 0\\ \sigma=\alpha\to\infty}} \lambda_{x(t)} = \varepsilon\lambda_2^{(N)} + o(\varepsilon) = \varepsilon\frac{\gamma^2}{8} + o(\varepsilon),$$

which is the same as that obtained in Refs. [18,19].

Typical plots of $\lambda_2^{(N)}$ are shown in Fig. 5 for $\sigma = 1.0$ and 2.0. The influence of the real noise excitation on the parametric resonance can be clearly seen. When σ increases, the influence of the harmonic excitation decreases and the effect of parametric resonance diminishes, resulting in the domination of the real noise excitation. To study the accuracy of $\lambda_2^{(N)}$ for different values of N, typical results of $\lambda_2^{(N)}$ are plotted in Fig. 6 for $N = 4$, 8, and 12. It is seen that $\lambda_2^{(8)}$ and $\lambda_2^{(12)}$ agree extremely well for all values of μ; whereas $\lambda_2^{(4)}$ agrees with $\lambda_2^{(8)}$ and $\lambda_2^{(12)}$ very well for smaller values of μ up to 6, but some discrepancies exist for larger values of μ.

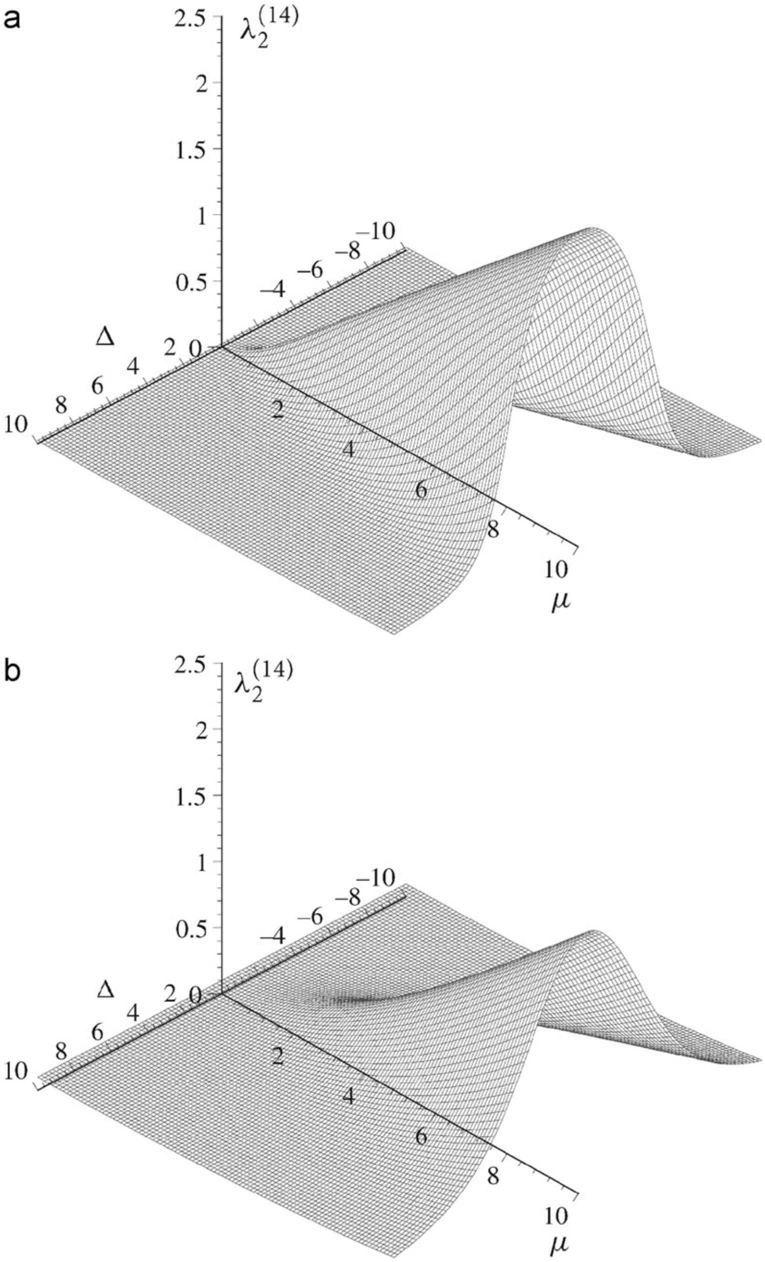

Fig. 5. Second-order perturbation of Lyapunov exponent $\lambda_2^{(14)}$, primary resonance, $\gamma = 1.0$ and $\alpha = 1.0$. (a) $\sigma = 1.0$ and (b) $\sigma = 2.0$.

In order to check the validity of the perturbation results, a digital simulation is performed. Eqs. (8) and (9) are discretized by using the Euler scheme and the resulting equations are:

$$\begin{aligned} x_1(t+\Delta t) &= x_1(t) + x_2(t)\cdot\Delta t, \\ x_2(t+\Delta t) &= x_2(t) - [1 + \varepsilon\mu\sin\nu t + \varepsilon^{1/2}\gamma\zeta(t)]x_1(t)\cdot\Delta t, \\ \zeta(t+\Delta t) &= (1-\alpha\cdot\Delta t)\zeta(t) + \sigma\cdot\Delta W(t), \end{aligned} \tag{31}$$

where $\Delta W(t) = n_t\sqrt{\Delta t}$, in which n_t is a standard normally distributed random number. Note that the Euler (31) is the same as the Milstein scheme. The numerical algorithm proposed by Wolf, Swift, Swinney and Vastano [20] for evaluating the Lyapunov exponents of a time series is applied to determine $\lambda_{x(t)}$. Numerical

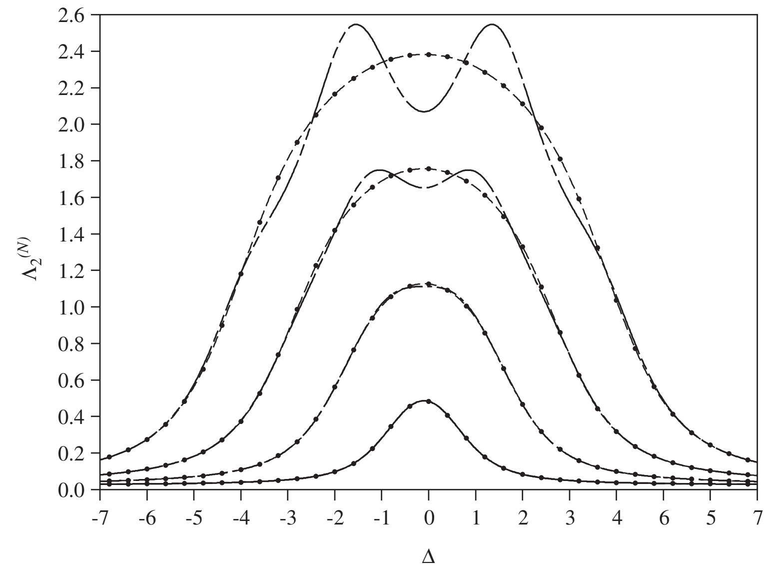

Fig. 6. Second-order perturbation of Lyapunov exponent $\lambda_2^{(N)}$, primary resonance, $\gamma = 1.0$, $\alpha = 1.0$, $\sigma = 1.0$. ●●●, $N = 12$; - - -, $N = 8$; —, $N = 4$, $\mu = 2.5$; ·— · —·, $N = 4$, $\mu = 5.0$; -———-, $N = 4$, $\mu = 7.5$; and - -———- -, $N = 4$, $\mu = 10.0$.

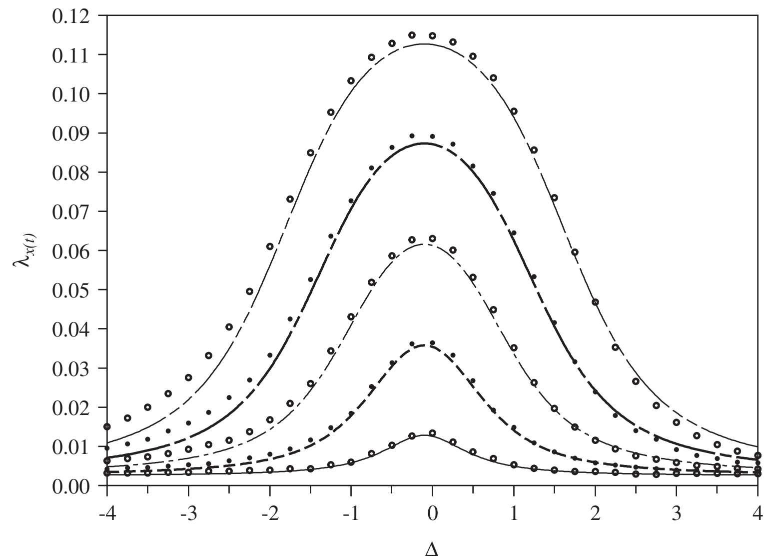

Fig. 7. Lyapunov exponent $\lambda_{x(t)}$, primary resonance, $\gamma = 1.0$, $\alpha = 1.0$, $\sigma = 1.0$, $\varepsilon = 0.1$. Dots (○ or ●), simulation; lines, analytical results, $N = 14$; —, $\mu = 1.0$; - - -, $\mu = 2.0$; ·— · —·, $\mu = 3.0$; -———-, $\mu = 4.0$; and - -———- -, $\mu = 5.0$.

results of $\lambda_{x(t)}$ from simulation are plotted in Fig. 7 along with the perturbation results $\lambda_{x(t)} \approx \varepsilon\lambda_2^{(14)}$ for various values of μ. It is observed that both results agree very well.

3.2. Secondary parametric resonance, $n = 2$, $\rho = 1$, and $v_0 = 1$

In the absence of the real noise excitation, the system described in Eq. (8) is in secondary parametric resonance when $v = v_0 + \varepsilon^2 \Delta$, $v_0 = 1$. By applying the transformation $\theta = \psi - \varphi$, the infinitesimal differential operator in Eq. (12) becomes

$$\mathscr{L}(p) = L_0 + \varepsilon L_1 + \varepsilon^2 L_2, \tag{32}$$

where

$$L_0 = \frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2} - \alpha\zeta\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \varphi},$$
$$L_1 = [-\gamma\zeta + \mu\sin(\varphi - \psi)]\cdot\left[\cos^2\varphi\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\varphi}\right) + p\cos\varphi\sin\varphi\right],$$
$$L_2 = \Delta\frac{\partial}{\partial\psi}.$$

Expand the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ and the eigenfunction $T(\zeta, \varphi, \psi)$:

$$\Lambda_{x(t)}(p) = \sum_{i=0}^{\infty}\varepsilon^i\Lambda_i(p), \quad T(\zeta, \varphi, \psi) = \sum_{i=0}^{\infty}\varepsilon^i T_i(\zeta, \varphi, \psi), \tag{33}$$

where $T_i(\zeta, \varphi, \psi)$ are periodic functions in φ and ψ of period 2π, respectively. Substituting Eq. (33) into Eq. (12) with the infinitesimal differential operator $\mathscr{L}(p)$ given by Eq. (32), expanding, and equating terms of equal power of ε results in the perturbation equations of the form given in Eq. (21).

Solving the perturbation equations, it is shown in Appendix C that $\Lambda_0(p) = \Lambda_1(p) = 0$ and $\Lambda_2(p)$ is the eigenvalue of the eigenvalue problem of Eq. (C.4) with a second-order ordinary differential operator, i.e.

$$a\ddot{\Psi}_0(\psi) + (b + q\cos 2\psi)\dot{\Psi}_0(\psi) + [\Lambda_2 + cp(p+2) + pq\sin 2\psi]\Psi_0(\psi) = 0, \tag{34}$$

where the coefficients a, b, c, and q are given in Appendix C.

3.2.1. Determination of Λ_2

The approach employed in Section 3.1.2 can be applied to determine Λ_2 by solving Eq. (C.4) in Appendix. Since the coefficients of system described by Eq. (C.4) are periodic functions of period π, a series expansion of the eigenfunction $\Psi_0(\psi)$ can be taken as

$$\Psi_0(\psi) = C_0 + \sum_{k=1}^{N}(C_{2k}\cos 2k\psi + S_{2k}\sin 2k\psi), \tag{35}$$

where C_0, C_{2k}, S_{2k}, $k = 1, 2, \ldots, N$ are coefficients to be determined. By substituting Eq. (35) into Eq. (C.4), multiplying by $\cos 2m\psi$, $\sin 2m\psi$, $m = 0, 1, \ldots, N$, respectively, and integrating with respect to ψ from 0 to π leads to a system of $2N + 1$ homogeneous linear algebraic equations for the unknown coefficients C_0, C_{2k}, S_{2k}, $k = 1, 2, \ldots, N$. The polynomial Eq. (24) is required in order to have non-trivial solutions for the unknown coefficients.

When $N = 1$, the coefficients of the cubic Eq. (24) are

$$d_0 = c^3p^6 + 6c^3p^5 + (-8ac^2 + 12\,c^3 - \tfrac{1}{2}cq^2)p^4 + (-32ac^2 + 8c^3 - 2cq^2)p^3$$
$$+ (16a^2c - 32ac^2 + 2aq^2 + 4b^2c - 2cq^2)p^2 + (32a^2c + 4aq^2 + 8b^2c)p,$$
$$d_1 = 3c^2p^4 + 12c^2p^3 + (-16ac + 12c^2 - \tfrac{1}{2}q^2)p^2 + (-32ac - q^2)p + 16a^2 + 4b^2,$$
$$d_2 = 3cp^2 + 6cp - 8a.$$

The analytical expression for $\Lambda_2^{(1)}$ is given by Eq. (25). In Fig. 8, the analytical results for $\Lambda_2^{(1)}$ are plotted along with the numerical results of $\Lambda_2^{(8)}$. Three-dimensional surface plots of $\Lambda_2^{(8)}$ are shown in Fig. 9 for $\sigma = 0.5$, 1.0, and 1.5 to illustrate the influence of the real noise excitation on the secondary parametric resonance.

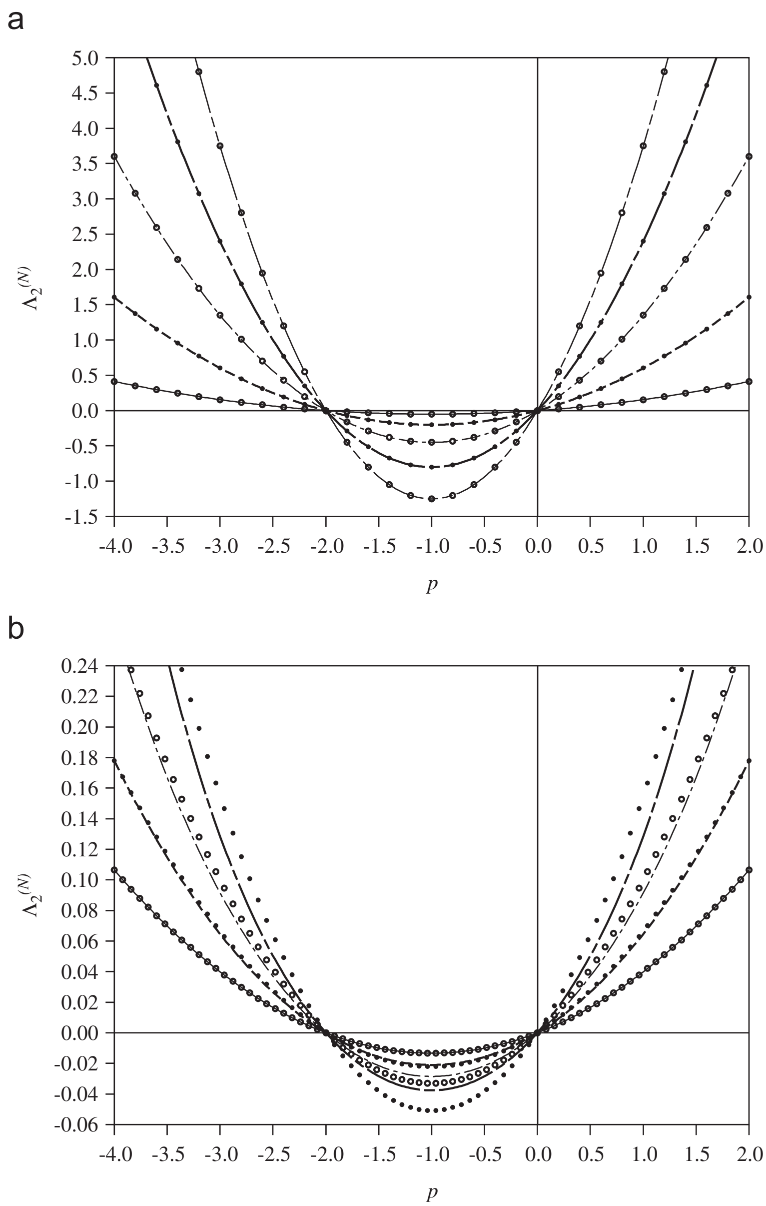

Fig. 8. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(N)}$, secondary resonance, $\Delta = 1.0$, $\gamma = 1.0$, $\alpha = 1.0$. Lines, $N = 1$, analytical results; dots (○ or ●), $N = 8$, numerical results: (a) $\mu = 1.0$; —, $\sigma = 2.0$; - - -, $\sigma = 4.0$; ·— · —·, $\sigma = 6.0$; -———-, $\sigma = 8.0$ and - -———- -, $\sigma = 10.0$; and (b) $\sigma = 1.0$; —, $\mu = 1.0$; - - -, $\mu = 2.0$; ·— · —·, $\mu = 2.5$; -———-, and $\mu = 3.0$.

The second-order perturbation of the Lyapunov exponent λ_2 can be determined by using Eq. (28). When $N = 8$, $\lambda_2^{(8)} = -2N^{(8)}/D^{(8)}$, where the expressions of $N^{(8)}$ and $D^{(8)}$ are given in Appendix D. Expressions of $\lambda_2^{(N)}$ for larger values of N are also obtained but are not presented here. Three-dimensional surface plots of $\lambda_2^{(14)}$ are shown in Fig. 10 for $\sigma = 1.0$ and 3.0. The impact of the real noise excitation on the secondary parametric resonance can be clearly seen. Similar conclusions on the qualitative behaviour of the moment Lyapunov exponent as in the case of primary parametric resonance can be drawn. It is noted that when $\Delta \to \pm\infty$ or $\mu \to 0$, Eqs. (26) and (30) are obtained, because in both cases the harmonic excitation has no effect and system of Eq. (8) is equivalent to a two-dimensional system under only real noise excitation when the stability is concerned.

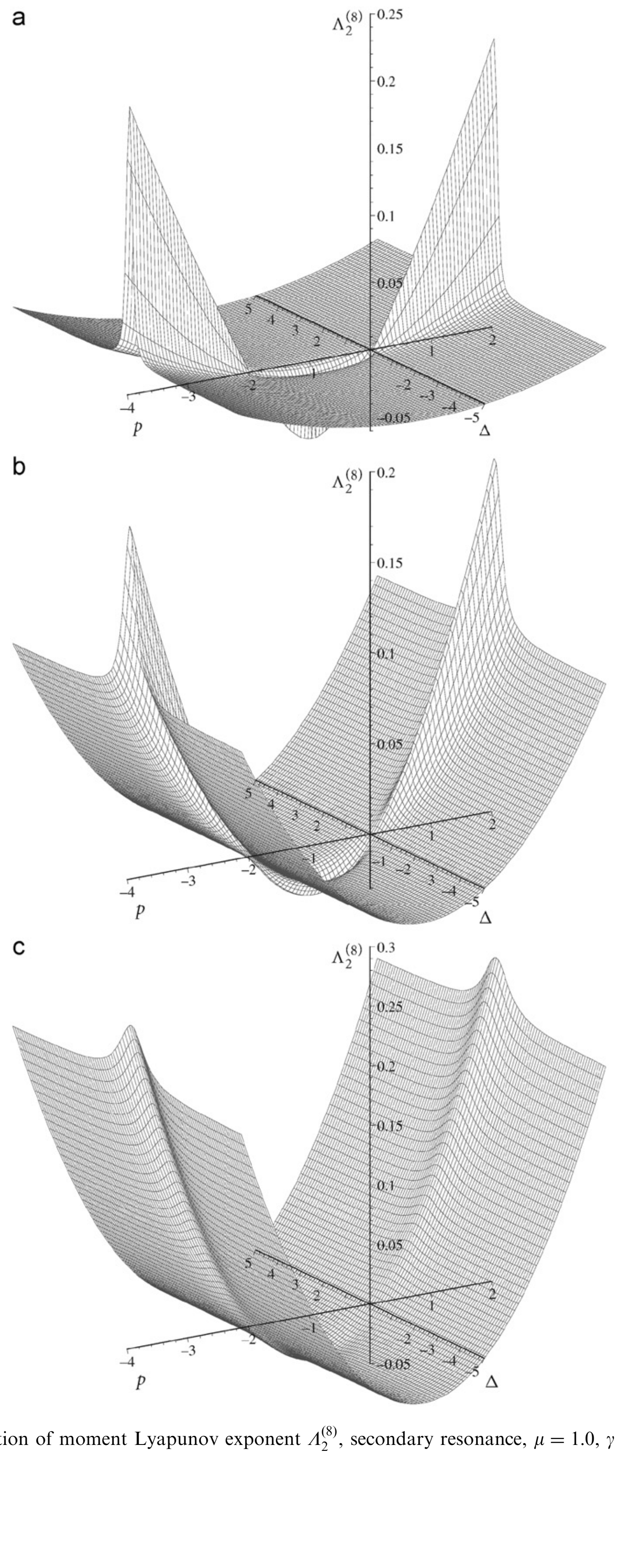

Fig. 9. Second-order perturbation of moment Lyapunov exponent $\Lambda_2^{(8)}$, secondary resonance, $\mu = 1.0$, $\gamma = 1.0$ and $\alpha = 1.0$. (a) $\sigma = 0.5$; (b) $\sigma = 1.0$ and (c) $\sigma = 1.5$.

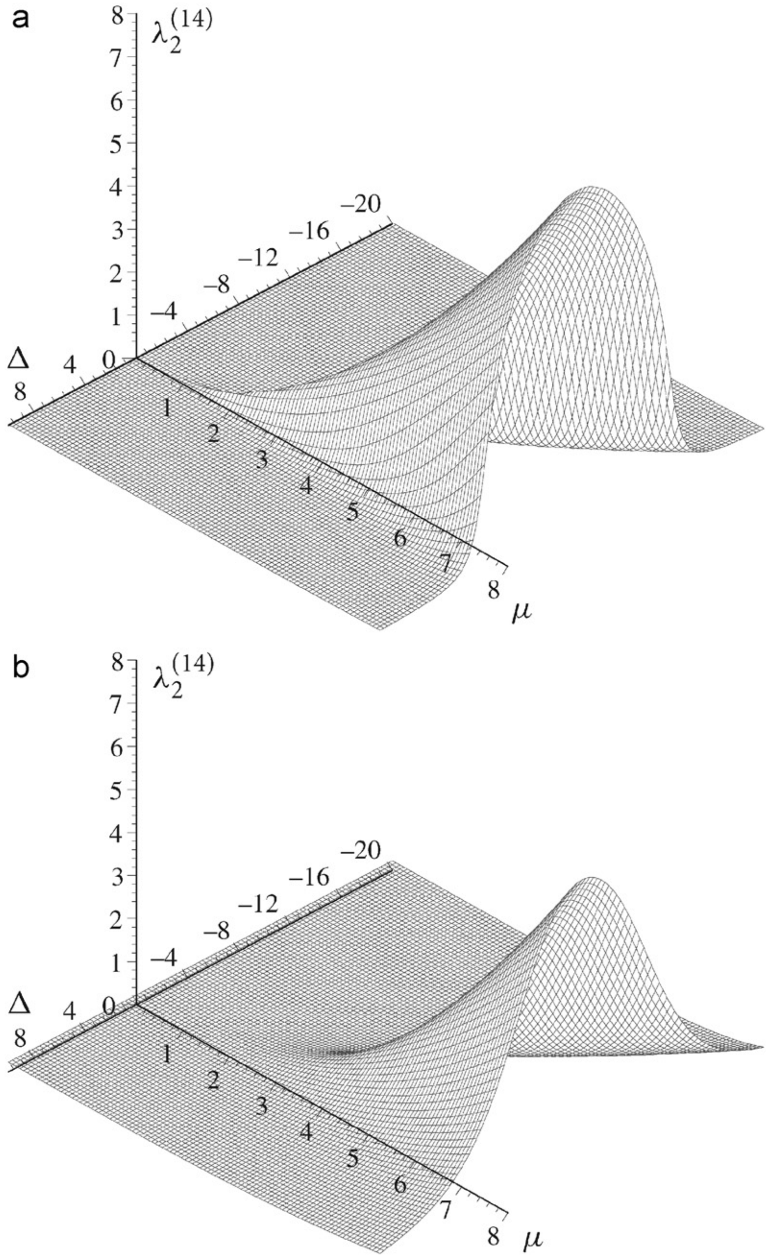

Fig. 10. Second-order perturbation of Lyapunov exponent $\lambda_2^{(14)}$, secondary resonance, $\gamma = 1.0$ and $\alpha = 1.0$. (a) $\sigma = 1.0$ and (b) $\sigma = 3.0$.

To study the accuracy of $\lambda_2^{(N)}$, typical values of $\lambda_2^{(N)}$ are plotted in Fig. 11 for $N = 4$, 8, and 12. It is seen that $\lambda_2^{(4)}$ yields accurate results for smaller values of μ up to 5, while $\lambda_2^{(8)}$ and $\lambda_2^{(12)}$ both yield accurate results for all values of μ shown. The validity of the perturbation results is checked by doing a Monte Carlo simulation, with the numerical results shown inFig. 12. It can be seen that the analytical results $\lambda_{x(t)} = \varepsilon^2 \lambda_2^{(14)}$ agree with $\lambda_{x(t)}$ obtained from simulation very well.

The Lyapunov exponent plots as seen in Figs. 5–7 are almost symmetric about $\Delta = 0$ in the primary parametric resonance case, whereas they are skewed towards $-\Delta$ in the secondary parametric resonance case as shown in Figs. 10–12.

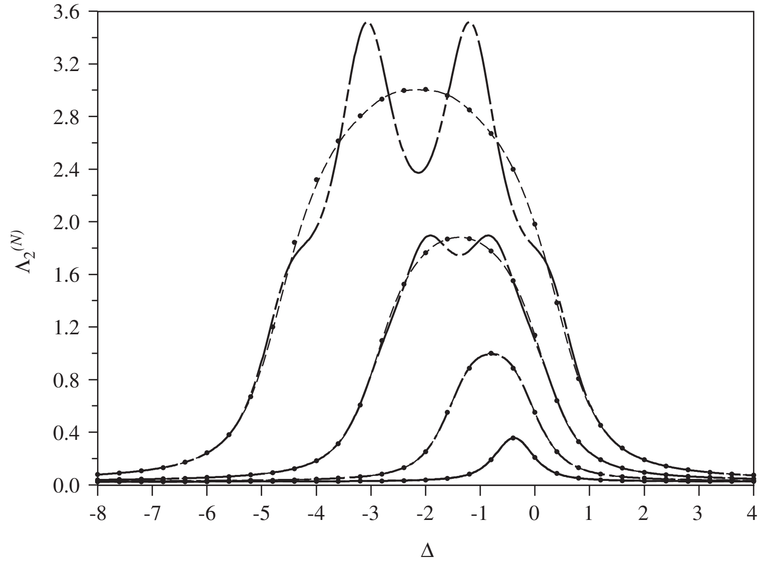

Fig. 11. Second-order perturbation of Lyapunov exponent $\lambda_2^{(N)}$, secondary resonance, $\gamma = 1.0$, $\alpha = 1.0$ and $\sigma = 1.0$. ●●●, $N = 12$; - - -, $N = 8$; —, $N = 4$, $\mu = 2.0$; ·— · —·, $N = 4$, $\mu = 3.0$; -——-, $N = 4$, $\mu = 4.0$; and - -——- -, $N = 4$, $\mu = 5.0$.

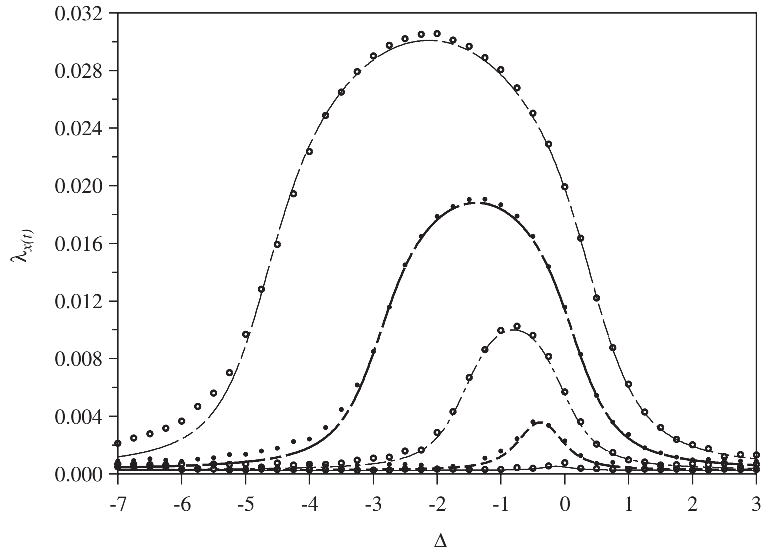

Fig. 12. Lyapunov exponent $\lambda_{x(t)}$, secondary resonance, $\gamma = 1.0$, $\alpha = 1.0$, $\sigma = 1.0$, $\varepsilon = 0.1$. Dots (○ or ●), simulation; lines, analytical results, $N = 14$; —, $\mu = 1.0$; - - -, $\mu = 2.0$; ·— · —·, $\mu = 3.0$; -——-, $\mu = 4.0$; and - -——- -, $\mu = 5.0$.

4. Conclusions

In this paper, the dynamic stability of a two-dimensional system under the parametric excitation of combined harmonic and real noise excitations is studied through the determination of the moment Lyapunov exponents and the Lyapunov exponents. The real noise, modelled as an Ornstein–Uhlenbeck process, is a more realistic model of noise in engineering applications than white noise. The eigenvalue problem governing the moment Lyapunov exponent is established by using the theory of stochastic dynamical systems. A regular perturbation method is employed to obtain expansions of the moment Lyapunov exponents. The Lyapunov exponents are determined by using the relationship between the moment Lyapunov exponents and the Lyapunov exponents. The accuracy of the expansions are studied and the validity of the expansions are checked by using Monte Carlo simulation.

The cases of both the primary and secondary parametric resonance in the absence of the real noise excitation are considered. The effect of the real noise excitation on the parametric resonance due to the harmonic excitation is studied. When the influence of the real noise excitation is small, the harmonic excitation is dominant and the parametric resonance is significant. On the other hand, when the impact of the real noise excitation is large, the effect of the harmonic excitation is small and the prominence of the parametric resonance due to the harmonic excitation diminishes.

In the special case when the amplitude of the harmonic excitation $\mu = 0$ or when the effect of the parametric resonance diminishes with the mistune $\Delta \to \pm\infty$, the moment Lyapunov exponent and the Lyapunov exponent reduce to those of a two-dimensional system under only the real noise excitation. Unfortunately, because of the current formulation is based on a stochastic dynamical system approach, it is not possible to set $\gamma = 0$ to reduce the result of the Lyapunov exponent to that of the Mathieu's equation, i.e. a two-dimensional system under harmonic excitation.

Acknowledgment

The research for this paper was supported, in part, by the Natural Sciences and Engineering Research Council of Canada. The author is grateful to the referee for the constructive comments which helped to improve the paper.

Appendix A. Perturbation Analysis—Primary resonance

A.1. Zeroth-order perturbation

The zeroth-order perturbation equation of Eq. (21) is $L_0 T_0 = \Lambda_0 T_0$, or

$$\frac{\sigma^2}{2}\frac{\partial^2 T_0}{\partial \zeta^2} - \alpha\zeta \frac{\partial T_0}{\partial \zeta} - \frac{\partial T_0}{\partial \varphi} = \Lambda_0 T_0. \tag{A.1}$$

It is well known that one of the properties of the moment Lyapunov exponent is $\Lambda_{x(t)}(0) = 0$, which implies $\Lambda_i(0) = 0$, $i = 0, 1, \ldots$. Because Eq. (A.1) does not contain the parameter p explicitly, if $\Lambda_0(0) = 0$ then $\Lambda_0(p) = 0$ for all values of p.

Seeking a solution of Eq. (A.1) in the form $T_0(\zeta, \varphi, \psi) = Z_0(\zeta)\Phi_0(\varphi)\Psi_0(\psi)$ and substituting into Eq. (A.1) leads to

$$\frac{\sigma^2}{2}\frac{\ddot{Z}_0}{Z_0} - \alpha\zeta\frac{\dot{Z}_0}{Z_0} = \frac{\dot{\Phi}_0}{\Phi_0} = k.$$

The Φ_0 equation results in $\Phi_0(\varphi) = C_1 e^{k\varphi}$. For $\Phi_0(\varphi)$ to be a periodic function in φ, it is required that $k = 0$, yielding $\Phi_0(\varphi) = C_1$. From the Z_0 equation, $Z_0(\zeta) = C_2 + C_3 \mathrm{erf}(\mathrm{i}\sqrt{\alpha}\zeta/\sigma)$, where $\mathrm{erf}(\cdot)$ denotes the error function. For $Z_0(\zeta)$ to be a bounded function when $\zeta \to \pm\infty$, it is required that $C_3 = 0$ and hence $Z_0(\zeta) = C_2$. Therefore, $T_0(\zeta, \varphi, \psi) = \Psi_0(\psi)$, where $\Psi_0(\psi)$ is a periodic function of period 2π.

The adjoint equation of Eq. (A.1) is

$$\frac{\sigma^2}{2}\frac{\partial^2 T_0^*}{\partial \zeta^2} + \alpha\zeta\frac{\partial T_0^*}{\partial \zeta} + \frac{\partial T_0^*}{\partial \varphi} + \alpha T_0^* = 0. \tag{A.2}$$

Similarly, seeking a solution of the form $T_0^*(\zeta, \varphi, \psi) = Z_0^*(\zeta)\Phi_0^*(\varphi)\Psi_0^*(\psi)$ and substituting into Eq. (A.2) leads to

$$\frac{\sigma^2}{2}\frac{\ddot{Z}_0^*}{Z_0^*} + \alpha\zeta\frac{\dot{Z}_0^*}{Z_0^*} + \alpha = -\frac{\dot{\Phi}_0^*}{\Phi_0^*} = \kappa,$$

and $\Psi_0^*(\psi)$ can be taken as an arbitrary periodic function of period 2π. The Φ_0^* equation yields $\Phi_0^*(\varphi) = \text{constant}$, $\kappa = 0$. Since the coefficients of Eq. (19) are periodic functions in φ of period π, $\Phi_0^*(\varphi)$ may be taken as

$$\Phi_0^*(\varphi) = 1/\pi, \quad 0 \leqslant \varphi < \pi, \tag{A.3}$$

which is the probability density function of a random number uniformly distributed in $(0, \pi)$.

The Z_0^* equation becomes

$$\frac{\sigma^2}{2}\ddot{Z}_0^* + \alpha\zeta\dot{Z}_0^* + \alpha Z_0^* = 0, \tag{A.4}$$

which is the Fokker–Planck equation for the stationary transition probability density of the Ornstein–Uhlenbeck process $\zeta(t)$ as defined in Eq. (9) [21]. Eq. (A.4) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\mathrm{d}Z_0^*}{\mathrm{d}\zeta} + \frac{2\alpha}{\sigma}\zeta Z_0^*\right) = 0,$$

or

$$\frac{\mathrm{d}Z_0^*}{\mathrm{d}\zeta} + \frac{2\alpha}{\sigma}\zeta Z_0^* = \text{probability current}. \tag{A.5}$$

Since both the probability density $Z_0^*(\zeta)$ and the probability current vanish when $\zeta \to \pm\infty$, Eq. (A.5) can be solved to yield

$$Z_0^*(\zeta) = \frac{1}{\sqrt{2\pi}\sigma_\zeta}\exp\left(-\frac{\zeta^2}{2\sigma_\zeta^2}\right), \tag{A.6}$$

where $\sigma_\zeta = \sigma/\sqrt{2\alpha}$ and $Z_0^*(\zeta)$ has been normalized using $\int_{-\infty}^{+\infty} Z_0^*(\zeta)\,\mathrm{d}\zeta = 1$. Eq. (A.6) indicates that the real noise process $\zeta(t)$, defined as an Ornstein–Uhlenbeck process, is a normally distributed random variable with mean value $\mu_\zeta = 0$ and standard deviation σ_ζ.

A.2. First-order perturbation

The first-order perturbation equation becomes, because $\Lambda_0(p) = 0$,

$$L_0 T_1 = \Lambda_1 T_0 - L_1 T_0. \tag{A.7}$$

Because $T_0(\zeta, \varphi, \psi) = \Psi_0(\psi)$, it is easy to show that

$$L_1 T_0 = -g_1^{(1)}(\varphi, \psi) \cdot \zeta,$$

where

$$g_1^{(1)}(\varphi, \psi) = \gamma[2\cos^2\varphi\dot{\Psi}_0(\psi) + p\cos\varphi\sin\varphi\Psi_0(\psi)].$$

The solvability condition of Eq. (A.7) is given by, from Fredholm Alternative $(\Lambda_1 T_0 - L_1 T_0, T_0^*) = 0$, where (f, g) denotes the inner product of functions $f(\zeta, \varphi, \psi)$ and $g(\zeta, \varphi, \psi)$ defined as

$$(f, g) = \int_{\psi=0}^{2\pi}\int_{\varphi=0}^{\pi}\int_{\zeta=-\infty}^{+\infty} f(\zeta, \varphi, \psi)g(\zeta, \varphi, \psi)\,\mathrm{d}\zeta\,\mathrm{d}\varphi\,\mathrm{d}\psi.$$

Hence,

$$\begin{aligned}\Lambda_1 &= \frac{(L_1 T_0, T_0^*)}{(T_0, T_0^*)} \\ &= \frac{1}{(T_0, T_0^*)}\{-\gamma\langle\zeta\rangle[2\overline{\cos^2\varphi}\langle\dot{\Psi}_0(\psi)\rangle + p\overline{\cos\varphi\sin\varphi}]\} = 0,\end{aligned}$$

in which $\langle f(\zeta)\rangle = \int_{-\infty}^{+\infty} f(\zeta)Z_0^*(\zeta)\,\mathrm{d}\zeta$ is the expected value of $f(\zeta)$, with ζ being a normally distributed random variable with mean zero and standard deviation σ_ζ, $\overline{g(\varphi)} = \int_0^\pi g(\varphi)\Phi_0^*(\varphi)\,\mathrm{d}\varphi$, and $\langle h(\psi)\rangle = \int_0^{2\pi} h(\psi)\Psi_0^*(\psi)\,\mathrm{d}\psi$.

The first-order perturbation equation becomes

$$L_0 T_1 = g_1^{(1)}(\varphi, \psi) \cdot \zeta. \tag{A.8}$$

From Xie [8], it is shown that a solution of the equation $L_0 T(\zeta, \varphi, \psi) = f(\zeta) g(\varphi, \psi)$ is given by

$$T(\zeta; \hat{\varphi}, s; \psi) = \int_0^s g(\hat{\varphi} - r, \psi) E[f(z(r))] \, \mathrm{d}r, \tag{A.9}$$

where $E[f(z(r))]$ is the expected value of the random variable $f(z(r))$, in which $z(r)$ is the normally distributed random variable with the mean value and variance given by

$$\mu_{z(r)} = \zeta \mathrm{e}^{-\alpha(r-s)}, \quad \sigma_{z(r)}^2 = \frac{\sigma^2}{2\alpha}[1 - \mathrm{e}^{-2\alpha(r-s)}].$$

The solution $T(\zeta, \varphi, \psi)$ is then obtained by replacing $\varphi = \hat{\varphi} - s$ and passing the limit $s \to -\infty$.

Hence, the solution of Eq. (A.8) is

$$T_1(\zeta, \varphi, \psi) = G_1^{(1)}(\varphi, \psi) \cdot \zeta,$$

where

$$\begin{aligned} G_1^{(1)}(\varphi, \psi) = & -\frac{\gamma}{2\alpha(\alpha^2 + 4)} [2(\alpha^2 \cos 2\varphi + 2\alpha \sin 2\varphi + \alpha^2 + 4) \dot{\Psi}_0(\psi) \\ & + p\alpha(\alpha \sin 2\varphi - 2 \cos 2\varphi) \Psi_0(\psi)]. \end{aligned}$$

A.3. Second-order perturbation

Since $\Lambda_0(p) = \Lambda_1(p) = 0$, the second-order perturbation equation is $L_0 T_2 = \Lambda_2 T_0 - L_1 T_1 - L_2 T_0$. From Fredholm Alternative, the solvability condition is $(\Lambda_2 T_0 - L_1 T_1 - L_2 T_0, T_0^*) = 0$. It is easy to show that

$$\begin{aligned} L_2 T_0 &= \Delta \dot{\Psi}_0(\psi) - \mu \sin(\psi - 2\varphi)[2 \cos^2 \varphi \dot{\Psi}_0(\psi) + p \cos \varphi \sin \varphi \Psi_0(\psi)], \\ L_1 T_1 &= -S_1^{(1)}(\varphi, \psi) \cdot \zeta - S_2^{(1)}(\varphi, \psi) \cdot \zeta^2, \end{aligned}$$

where

$$\begin{aligned} S_1^{(1)}(\varphi, \psi) &= \gamma[2 \cos^2 \varphi \dot{\Psi}_0(\psi) + p \cos \varphi \sin \varphi \Psi_0(\psi)], \\ S_2^{(1)}(\varphi, \psi) &= \gamma \left[\cos^2 \varphi \left(2 \frac{\partial G_1^{(1)}}{\partial \psi} + \frac{\partial G_1^{(1)}}{\partial \varphi} \right) + p \cos \varphi \sin \varphi G_1^{(1)} \right]. \end{aligned}$$

The solvability condition becomes

$$\int_{\psi=0}^{2\pi} \int_{\varphi=0}^{\pi} \int_{\zeta=-\infty}^{+\infty} (\Lambda_2 T_0 - L_1 T_1 - L_2 T_0) \, Z_0^*(\zeta) \Phi_0^*(\varphi) \Psi_0^*(\psi) \, \mathrm{d}\zeta \, \mathrm{d}\varphi \, \mathrm{d}\psi = 0,$$

or

$$\begin{aligned} \int_{\psi=0}^{2\pi} \Bigg\{ \int_{\varphi=0}^{\pi} & [\Lambda_2 \Psi_0(\psi) + \langle \zeta^2 \rangle S_2^{(1)}(\varphi, \psi) - \Delta \dot{\Psi}_0(\psi) \\ & + \mu \sin(\psi - 2\varphi)(2 \cos^2 \varphi \dot{\Psi}_0(\psi) + p \cos \varphi \sin \varphi \Psi_0(\psi))] \, \mathrm{d}\varphi \Bigg\} \Psi_0^*(\psi) \, \mathrm{d}\psi = 0. \end{aligned} \tag{A.10}$$

Since Eq. (A.10) is valid for an arbitrary function $\Psi_0^*(\psi)$, the solvability condition leads to

$$\begin{aligned} \int_{\varphi=0}^{\pi} & [\Lambda_2 \Psi_0(\psi) + \langle \zeta^2 \rangle S_2^{(1)}(\varphi, \psi) - \Delta \dot{\Psi}_0(\psi) \\ & + \mu \sin(\psi - 2\varphi)(2 \cos^2 \varphi \dot{\Psi}_0(\psi) + p \cos \varphi \sin \varphi \Psi_0(\psi))] \, \mathrm{d}\varphi = 0, \end{aligned}$$

or

$$a\ddot{\Psi}_0(\psi) + (b + 2q\sin\psi)\dot{\Psi}_0(\psi) + [\Lambda_2 + cp(p+2) - pq\cos\psi]\Psi_0(\psi) = 0, \tag{A.11}$$

where

$$a = -\frac{\gamma^2\sigma^2(3\alpha^2+8)}{4\alpha^2(\alpha^2+4)}, \quad b = -\frac{8\alpha\Delta + 2\alpha^3\Delta + \gamma^2\sigma^2}{2\alpha(\alpha^2+4)}, \quad c = -\frac{\gamma^2\sigma^2}{16(\alpha^2+4)}, \quad q = \frac{\mu}{4}.$$

Hence, the second-order perturbation of the moment Lyapunov exponent $\Lambda_2(p)$ is the eigenvalue of the eigenvalue problem described in Eq. (A.11) with a second-order ordinary differential operator, in which $\Psi_0(\psi)$ is the associated eigenfunction.

Appendix B. $\lambda_2^{(8)}$ in the primary parametric resonance region

$$\lambda_2^{(8)} = -\frac{N^{(8)}}{D^{(8)}}, \tag{B.1}$$

$$\begin{aligned} N^{(8)} = {} & 2(2a+c)q^{16} + 10(520a^3 + 80a^2c + ab^2 - 4b^2c)q^{14} + 10(36\,312a^5 + 10\,160a^4c \\ & + 237a^3b^2 + 106a^2b^2c - 3ab^4 + 26\,b^4c)q^{12} + (10\,249\,920a^7 + 4\,461\,120a^6c \\ & + 231\,524a^5b^2 + 59\,888a^4b^2c + 1555a^3b^4 - 5540a^2b^4c + 71ab^6 - 628b^6c)q^{10} \\ & + 2(66\,427\,200a^9 + 43\,352\,640a^8c + 3\,631\,388a^7b^2 + 1\,921\,132a^6b^2c + 42\,147a^5b^4 \\ & + 65\,583a^4b^4c - 798a^3b^6 + 8178a^2b^6c - 37ab^8 + 367b^8c)q^8 + (16a^2+b^2) \\ & \times(25a^2+b^2)\{(1\,998\,864a^7 + 1\,985\,760a^6c + 26\,256a^5b^2 - 21\,400a^4b^2c + 591a^3b^4 \\ & - 1700a^2b^4c + 39ab^6 - 460b^6c)q^6 + (9a^2+b^2)(36a^2+b^2)\{2(7840a^5 + 12\,964a^4c \\ & - 85a^3b^2 - 493a^2b^2c - 5ab^4 + 79b^4c)q^4 + (4a^2+b^2)(49a^2+b^2)[(64a^3 + 224a^2c \\ & + ab^2 - 28b^2c)q^2 + 2c(a^2+b^2)(64a^2+b^2)]\}\}, \end{aligned} \tag{B.2}$$

$$\begin{aligned} D^{(8)} = {} & q^{16} + 20(20a^2 - b^2)q^{14} + 10(5080a^4 + 53a^2b^2 + 13b^4)q^{12} + 2(1\,115\,280a^6 \\ & + 14\,972a^4b^2 - 1385a^2b^4 - 157b^6)q^{10} + (43\,352\,640a^8 + 1\,921\,132a^6b^2 + 65\,583a^4b^4 \\ & + 8178a^2b^6 + 367b^8)q^8 + (16a^2+b^2)(25a^2+b^2)\{10(99\,288a^6 - 1070a^4b^2 - 85a^2b^4 \\ & - 23b^6)q^6 + (9a^2+b^2)(36a^2+b^2)\{(12\,964a^4 - 493a^2b^2 + 79b^4)q^4 + (4a^2+b^2) \\ & \times(49a^2+b^2)[14(8a^2-b^2)q^2 + (a^2+b^2)(64a^2+b^2)]\}\}. \end{aligned} \tag{B.3}$$

Appendix C. Perturbation analysis—secondary resonance

C.1. Zeroth-order perturbation

The zeroth-order perturbation equation is $L_0T_0 = \Lambda_0T_0$, which is the same as that in the case of the primary parametric resonance (Eq. (A.1)). Following the same procedure as in Section A.1, $\Lambda_0(p) = 0$ and $T_0(\zeta,\varphi,\psi) = \Psi_0(\psi)$, where $\Psi_0(\psi)$ is a periodic function of period 2π. The solution of the associated adjoint equation given by Eq. (A.2) is $T_0^*(\zeta,\varphi,\psi) = Z_0^*(\zeta)\Phi_0^*(\varphi)\Psi_0^*(\psi)$, where

$$\Phi_0^*(\varphi) = \frac{1}{2\pi}, \qquad 0 \leqslant \varphi < 2\pi, \tag{C.1}$$

which is the probability density function of a random number uniformly distributed in $(0, 2\pi)$, $Z_0^*(\zeta)$ is given by Eq. (A.6), and $\Psi_0^*(\psi)$ is an arbitrary periodic function of period 2π.

C.2. First-order perturbation

Since $\Lambda_0(p)=0$, the first-order perturbation equation becomes $L_0T_1=\Lambda_1T_0-L_1T_0$. The solvability condition is, from Fredholm Alternative, $(\Lambda_1T_0-L_1T_0,T_0^*)=0$, where (f,g) denotes the inner product of functions $f(\zeta,\varphi,\psi)$ and $g(\zeta,\varphi,\psi)$ defined as

$$(f,g)=\int_{\psi=0}^{2\pi}\int_{\varphi=0}^{2\pi}\int_{\zeta=-\infty}^{+\infty}f(\zeta,\varphi,\psi)g(\zeta,\varphi,\psi)\,\mathrm{d}\zeta\,\mathrm{d}\varphi\,\mathrm{d}\psi.$$

Because $T_0(\zeta,\varphi,\psi)=\Psi_0(\psi)$, one has

$$L_1T_0=[-\gamma\zeta+\mu\sin(\varphi-\psi)]\cdot[\cos^2\varphi\dot{\Psi}_0(\psi)+p\cos\varphi\sin\varphi\Psi_0(\psi)],$$

and the solvability condition leads to

$$\begin{aligned}\Lambda_1&=\frac{1}{(T_0,T_0^*)}\int_{\psi=0}^{2\pi}[-\gamma\langle\zeta\rangle+\mu\overline{\sin(\varphi-\psi)}]\cdot[\overline{\cos^2\varphi}\dot{\Psi}_0(\psi)+p\overline{\cos\varphi\sin\varphi}]\Psi_0^*(\psi)\,\mathrm{d}\psi,\\&=0,\end{aligned}$$

where $\langle f(\zeta)\rangle=\int_{-\infty}^{+\infty}f(\zeta)Z_0^*(\zeta)\,\mathrm{d}\zeta$ is the expected value of $f(\zeta)$, with $\zeta=N(0,\sigma_\zeta)$, and $\overline{g(\varphi)}=\int_0^{2\pi}g(\varphi)\Phi_0^*(\varphi)\,\mathrm{d}\varphi$.

The first-order perturbation equation becomes

$$L_0T_1=g_0^{(1)}(\varphi,\psi)+g_1^{(1)}(\varphi,\psi)\cdot\zeta, \tag{C.2}$$

where

$$\begin{aligned}g_0^{(1)}(\varphi,\psi)&=-\mu\sin(\varphi-\psi)[\cos^2\varphi\dot{\Psi}_0(\psi)+p\cos\varphi\sin\varphi\Psi_0(\psi)],\\g_1^{(1)}(\varphi,\psi)&=\gamma[\cos^2\varphi\dot{\Psi}_0(\psi)+p\cos\varphi\sin\varphi\Psi_0(\psi)].\end{aligned}$$

The solution of Eq. (C.2) can be obtained by using Eq. (A.9) and is

$$T_1(\zeta,\varphi,\psi)=G_0^{(1)}(\varphi,\psi)+G_1^{(1)}(\varphi,\psi)\cdot\zeta,$$

where

$$\begin{aligned}G_0^{(1)}(\varphi,\psi)=&\ \frac{\mu}{12}\{[4\cos\psi-4p\sin\psi+\cos(\varphi+\psi)-6\cos(\varphi-\psi)-\cos(3\varphi-\psi)]\dot{\Psi}_0(\psi)\\&+p[3\sin(\varphi+\psi)-\sin(3\varphi-\psi)]\Psi_0(\psi)\},\\G_1^{(1)}(\varphi,\psi)=&\ -\frac{\gamma}{2\alpha(\alpha^2+4)}[2(\alpha^2\cos2\varphi+2\alpha\sin2\varphi+\alpha^2+4)\dot{\Psi}_0(\psi)\\&+p\alpha(\alpha\sin2\varphi-2\cos2\varphi)\Psi_0(\psi)].\end{aligned}$$

C.3. Second-order perturbation

Since $\Lambda_0(p)=\Lambda_1(p)=0$, the second-order perturbation equation is reduced to $L_0T_2=\Lambda_2T_0-L_1T_1-L_2T_0$. It is easy to show that

$$\begin{aligned}L_1T_1&=\mu\sin(\varphi-\psi)S_0^{(1)}-[\gamma S_0^{(1)}-\mu\sin(\varphi-\psi)S_1^{(1)}]\cdot\zeta-\gamma S_1^{(1)}\cdot\zeta^2,\\L_2T_0&=\Delta\dot{\Psi}_0(\psi),\end{aligned}$$

where

$$S_i^{(1)}(\varphi,\psi)=\cos^2\varphi\left(\frac{\partial G_i^{(1)}}{\partial\varphi}+\frac{\partial G_i^{(1)}}{\partial\psi}\right)+p\cos\varphi\sin\varphi G_i^{(1)},\quad i=0,1.$$

The solvability condition of the second-order perturbation equation is, from Fredholm Alternative, $(\Lambda_2 T_0 - L_1 T_1 - L_2 T_0, T_0^*) = 0$, which leads to

$$\int_{\psi=0}^{2\pi}\left\{\int_{\varphi=0}^{2\pi}[\Lambda_0\Psi_0(\psi) - \mu\sin(\varphi-\psi)S_0^{(1)} + \gamma S_1^{(1)}\langle\zeta^2\rangle - \Delta\dot{\Psi}_0(\psi)]\,\mathrm{d}\varphi\right\}\Psi_0^*(\psi)\,\mathrm{d}\psi = 0. \tag{C.3}$$

Because Eq. (C.3) is valid for an arbitrary periodic function $\Psi_0^*(\psi)$, one must have

$$\int_{\varphi=0}^{2\pi}[\Lambda_0\Psi_0(\psi) - \mu\sin(\varphi-\psi)S_0^{(1)} + \gamma S_1^{(1)}\langle\zeta^2\rangle - \Delta\dot{\Psi}_0(\psi)]\,\mathrm{d}\varphi = 0,$$

which yields, after integration,

$$a\ddot{\Psi}_0(\psi) + (b + q\cos 2\psi)\dot{\Psi}_0(\psi) + [\Lambda_2 + cp(p+2) + pq\sin 2\psi]\Psi_0(\psi) = 0, \tag{C.4}$$

where

$$a = -\frac{\gamma^2\sigma^2(3\alpha^2+8)}{16\alpha^2(\alpha^2+4)},\quad b = -\frac{\alpha(\alpha^2+4)(\mu^2+12\Delta)+3\gamma^2\sigma^2}{12\alpha(\alpha^2+4)},\quad c = -\frac{\gamma^2\sigma^2}{16(\alpha^2+4)},\quad q = \frac{\mu^2}{8}.$$

Hence, the second-order perturbation of the moment Lyapunov exponent Λ_2 is given by the eigenvalue of system described by Eq. (C.4) with a second-order ordinary differential operator.

Appendix D. $\lambda_2^{(8)}$ in the secondary parametric resonance region

$$\lambda_2^{(8)} = -2\frac{N^{(8)}}{D^{(8)}}, \tag{D.1}$$

$$\begin{aligned}
N^{(8)} = {} & (40a+c)q^{16} + 80(2080a^3 + 80a^2c + ab^2 - b^2c)q^{14} + 160(1\,161\,984a^5 + 81\,280a^4c \\
& + 1896a^3b^2 + 212a^2cb^2 - 6ab^4 + 13b^4c)q^{12} + 128(655\,994\,880a^7 + 71\,377\,920a^6c \\
& + 3\,704\,384a^5b^2 + 239\,552a^4b^2c + 6220a^3b^4 - 5540a^2b^4c + 71ab^6 - 157\,b^6c)q^{10} \\
& + 256(68\,021\,452\,800a^9 + 11\,098\,275\,840a^8c + 929\,635\,328a^7b^2 + 122\,952\,448a^6b^2c \\
& + 2\,697\,408a^5b^4 + 1\,049\,328a^4b^4c - 12\,768a^3b^6 + 32\,712a^2b^6c - 148ab^8 + 367b^8c)q^8 \\
& + 2048(64a^2+b^2)(100a^2+b^2)\{(127\,927\,296a^7 + 31\,772\,160a^6c + 420\,096a^5b^2 \\
& - 85\,600a^4b^2c + 2364a^3b^4 - 1700a^2b^4c + 39ab^6 - 115\,b^6c)q^6 + 2(36a^2+b^2) \\
& \times(144a^2+b^2)\{(501\,760a^5 + 207\,424a^4c - 1360a^3b^2 - 1972a^2b^2c - 20ab^4 \\
& + 79b^4c)q^4 + 8(16a^2+b^2)(196a^2+b^2)[(256a^3 + 224a^2c + ab^2 - 7b^2c)q^2 \\
& + 2c(4a^2+b^2)(256a^2+b^2)]\}\},
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
D^{(8)} = {} & q^{16} + 80\,(80a^2 - b^2)q^{14} + 160(81\,280a^4 + 212a^2b^2 + 13b^4)q^{12} \\
& + 128(71\,377\,920a^6 + 239\,552a^4b^2 - 5540a^2b^4 - 157b^6)q^{10} \\
& + 256(11\,098\,275\,840a^8 + 122\,952\,448a^6b^2 + 1\,049\,328a^4b^4 + 32\,712a^2b^6 + 367b^8)q^8 \\
& + 2048(64a^2+b^2)(100a^2+b^2)\{5(6\,354\,432a^6 - 17\,120a^4b^2 - 340a^2b^4 - 23b^6)q^6 \\
& + 2(36a^2+b^2)(144a^2+b^2)\{(207\,424a^4 - 1972a^2b^2 + 79b^4)q^4 \\
& + 8(16a^2+b^2)(196a^2+b^2)[7(32a^2-b^2)q^2 + 2(4a^2+b^2)(256a^2+b^2)]\}\}.
\end{aligned} \tag{D.3}$$

References

[1] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998.

[2] L. Arnold, E. Oeljeklaus, E. Pardoux, Almost sure and moment stability for linear Itô equations, in: L. Arnold, V. Wihstutz (Eds.), *Lyapunov Exponents*, *Lecture Notes in Mathematics*, Vol. 1186, Springer, Berlin, 1986, pp. 85–125.

[3] W. Wedig, Lyapunov exponent of stochastic systems and related bifurcation problems, in: S.T. Ariaratnam, G.I. Schuëller, I. Elishakoff (Eds.), *Stochastic Structural Dynamics—Progress in Theory and Applications*, Elsevier Applied Science, Amsterdam, 1988, pp. 315–327.

[4] L. Arnold, M.M. Doyle, N. Sri Namachchivaya, Small noise expansion of moment Lyapunov exponents for two-dimensional systems, *Dynamics and Stability of Systems* 12 (3) (1997) 187–211.

[5] R. Khasminskii, N. Moshchuk, Moment Lyapunov exponent and stability index for linear conservative system with small random perturbation, *SIAM Journal of Applied Mathematics* 58 (1) (1998) 245–256.

[6] N. Sri Namachchivaya, L. Vedula, Stabilization of linear systems by noise: application to flow induced oscillations, *Dynamics and Stability of Systems* 15 (2) (2000) 185–208.

[7] N. Sri Namachchivaya, H.J. Van Roessel, Moment Lyapunov exponent and stochastic stability of two coupled oscillators driven by real noise, *Journal of Applied Mechanics* 68 (6) (2001) 903–914.

[8] W.-C. Xie, Moment Lyapunov exponents of a two-dimensional system under real noise excitation, *Journal of Sound and Vibration* 239 (1) (2001) 139–155.

[9] W.-C. Xie, Moment Lyapunov exponents of a two-dimensional system under bounded noise parametric excitation, *Journal of Sound and Vibration* 263 (3) (2003) 593–616.

[10] N. Sri Namachchivaya, Almost sure stability of dynamical systems under combined harmonic and stochastic excitations, *Journal of Sound and Vibration* 151 (1) (1991) 77–90.

[11] P. Baxendale, Lyapunov exponents and resonance for small periodic and random perturbations of a conservative system, *Stochastics and Dynamics* 2 (2002) 49–66.

[12] W.-C. Xie, Moment Lyapunov exponents of a two-dimensional system under both harmonic and white noise excitations, *Journal of Sound and Vibration* 289 (1–2) (2006) 171–191.

[13] W.-C. Xie, *Dynamic Stability of Structures*, Cambridge University Press, New York, 2006.

[14] R.Z. Khasminskii, On stochastic processes defined by differential equations with a small parameter, *Theory of Probability and Its Applications* 11 (2) (1966) 211–228 (English Translation).

[15] R.Z. Khasminskii, A limit theorem for the solutions of differential equations with random right-hand sides, *Theory of Probability and Its Applications* 11 (3) (1966) 390–406 (English Translation).

[16] R.Z. Khasminskii, Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, *Theory of Probability and Its Applications* 12 (1967) 144–147 (English Translation).

[17] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, second ed., Wiley, New York, 1989.

[18] E. Pardoux, V. Wihstutz, Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion, *SIAM Journal of Applied Mathematics* 48 (2) (1988) 442–457.

[19] S.T. Ariaratnam, W.-C. Xie, Lyapunov exponents and stochastic bifurcations, in: W. Schiehlen (Ed.), *Proceedings of IUTAM Symposium on Nonlinear Dynamic Engineering Systems*, Stuttgart, Germany, August 21–25, Springer, Berlin, 1989, pp. 1–8.

[20] A. Wolf, J. Swift, H. Swinney, A. Vastano, Determining Lyapunov exponents from a time series, *Physica D* 16 (1985) 285–317.

[21] H. Risken, *The Fokker–Planck Equation: Methods of Solution and Applications*, second ed., Springer, Berlin, 1989.